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Indifference valuation in non-reduced incomplete models with a stochastic risk factor

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Indifference valuation in non-reduced incomplete models with a stochastic risk factor

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To my parents, Sergey Sokolov and Inessa Sokolova

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Indifference valuation in non-reduced incomplete models with a stochastic risk factor

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This work contributes to the methodology of valuation of financial derivative contracts in an incomplete market. It focuses on a special type of incompleteness caused by the presence of a non-traded stochastic risk factor, affecting the value of the contract. The non-traded risk factor may only appear in the payoff of the contract or, in addition, may enter the dynamics of the traded asset. We consider both cases. We suggest a discrete time discrete space binomial model for the traded stock and the non-traded risk factor. We work in the utility maximization framework with dynamically changing agent's preferences. We present a discrete time multi-period analog of the forward and backward utility processes recently developed in continuous time. We use methods of stochastic control and provide the indifference valuation algo-

rithm with both the forward and backward dynamic utilities. We compare the two approaches and provide conditions under which they assign the same value to the contract. We show that unlike the backward dynamic utility, the forward dynamic utility yields prices that do not depend on the end of the investment horizon. We pay attention to the choice of the equivalent martingale measure used for valuation (i.e., the minimal martingale measure and the minimal entropy measure for the forward and the backward utility processes correspondingly). We explicitly characterize both measures and give conditions under which they coincide. We extend our algorithm to the case of American and partial exercise contracts. We illustrate our work with numerical examples, showing that in an incomplete market, a call option on a non-traded risk factor may optimally be exercised early, and that it may be optimal to exercise only a fraction of the total number of contracts held, if partial exercise is allowed. In continuous time we extend the existing results to the case of American contracts with both the backward and the forward utilities. We emphasize the similarities between our discrete time valuation algorithm and the continuous time valuation. The two approaches use the same pricing measures, yield prices through nonlinear functionals of similar form, exhibit a similar relationship between the backward and forward prices, and a similar structure for the aggregate minimal entropy. We believe that our work makes a contribution by exposing the two above mentioned ways of dependence on the non-traded risk factor, and by providing a new dynamic indifference pricing algorithm that allows consistent valuation across different investment horizons.

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Chapter 1

Introduction

This work contributes to the methods used for valuation of financial derivative contracts. It considers the case where the value of the contract depends on both the value of the underlying exchange-traded asset and the level of a non-traded risk factor. In a classical model, where neither the contract's payoff nor the traded asset's dynamics depends on the risk factor, the well-known arbitrage-free valuation theory suggests pricing the contract by calculating the risk-neutral expectation of the contract's discounted stochastic payoff. Assuming that the traded asset follows a lognormal stochastic process, the famous Black-Scholes formula ([5], [37]) applies. The risk factor is usually assumed to follow a stochastic process, correlated to the traded asset process. Not only the payoff of the contract, but also the dynamics of the traded asset may depend explicitly on the past and current levels of the non-traded risk factor. The non-traded factor carries extra exogenous risk, which cannot be eliminated through trading with available market instruments. Thus, it becomes impossible to replicate exactly the payoff of the contract. Without the perfect replication assumption, the classical no-arbitrage valuation does not give a unique answer for the value of such a contract since, in this case, more than one risk-neutral measure exists. The market is then termed incomplete and other methods must to be applied to value those contracts that are contingent on the non-traded risk factor.

One way to extend the classical no-arbitrage theory for an incomplete market settings is to assume that the agent, trading in the market and holding the contingent contract, has pre-specified preferences regarding stochastic outcomes of his trading strategies and market dynamics. Depending on these preferences, each agent will then have his own price, which may differ from the price of other agents. The agent's preferences determine his attitude towards the risk carried by the non-traded stochastic factor. Specifying the agent's preferences leads to pricing using the so-called utility maximization approach, originated by [36]. Utility maximization can usually be addressed using either duality methods or the methods of dynamic programming and stochastic control. Duality methods allow one to treat even very general utility maximization problems, such those in which the Markovian property does not hold. However the results usually are not as explicit as when using the methods of stochastic control and dynamic programming developed by [30] and [15]. The latter are methods used in this work.

For utility maximization, an important question is which utility function (or preferences) to choose. A number of different choices have been suggested, such as a utility function of a quadratic form, for example. The latter leads to methods of local risk minimization and mean-variance pricing ([19] provide a summary and comparison of the two approaches). Quadratic utility function is also used in the ϵ - arbitrage approach (see [4]). A more general choice of a power utility function is used in [48], [33] and [21]. Another common choice is to assume the agent's utility of an exponential form, as it is done in this work. In the simplest case, the utility function can be static, i.e, it either has the same functional form at all times or is assigned and fixed at one particular time point only. In a more general setting, the agent's preferences may be dynamically changing with the market, as it is in this work.

There is a growing community of researchers who support the idea that an agent's preferences should be dynamic in order to capture changes in his attitude towards the environment changes. One way to extend the traditional static exponential utility, prescribed at only one future point in time, is to roll it backward recursively from the terminal time to the current valuation time. A rich class of utilities can be obtained in such a way. However, having a fixed utility at the end of the investment horizon is not a very desirable feature since it imposes the restriction that all the contracts valued must mature prior to the end of the investment horizon. If an additional contract is to be valued with a longer maturity, the end of the investment horizon needs to be changed and all the contracts re-evaluated accordingly. Doing so would create an artificial mispricing, a consequence of the utility function choice. In this work we present two different dynamic utility processes. One of these is derived backward in time, starting at the end of the investment horizon. The other progresses forward in time, and does not have the unattractive feature described above.

In addition to choosing a specific form of the utility function, one must also choose functional form of the stochastic processes describing the evolution of the traded stock and the non-traded factor. As mentioned above, two types of dependence are possible. With the first, the dynamics of the traded asset only depend on the traded asset itself, and the dependence on the non-traded risk factor only enters through the payoff of the contract. This type of dependence usually yields more explicit results and is commonly used in the literature. Examples of such models include [48], [20], and [18]. The other, more fundamental type of dependence arises when the traded asset's dynamics depend explicitly on the value of the risk factor. In such models the valuation procedure is usually more complex and is the subject of this work.

Throughout this thesis we work with two dynamic utilities. Our work is motivated by that of [42] in continuous time, which suggests two different dynamic utility processes, the backward and the forward. We construct the two similar utility processes in discrete time. In the absence of any derivative contracts, both utilities are invariant with respect to the end of the investment horizon, as in [42]. That is, starting with the same initial wealth, the optimal investment in the interval $[t; T_1)$ yields the same expected utility of terminal wealth as the one in the interval $[t; T_2)$. The backward utility is normalized (fixed) at a future time point T, and turns out to coincide with the plain investment value function V^0, considered in [41]. The forward utility introduced by [42] in continuous time is fixed at a prior time, not a future time. Is shown in this work, the forward utility does not impose any time restrictions on the maturities of valued contracts, and yields prices that do not depend on the end of the investment horizon. We use the indifference valuation framework to derive the corresponding prices and their properties first for European, then American, and later partial exercise contracts. We compare the forward valuation algorithm to the backward valuation algorithm for European and American contracts.

The holder can terminate (i.e., exercise) an American contract at any time within a pre-specified period, and receive the upon agreed cashflow, called the intrinsic payoff. A partial exercise contract can be viewed as a bundle of identical

American contracts, each of which can be exercised at a different time within the same pre-specified horizon. At any time the holder has to decide how many (what fraction) of all the American contracts held needs to be exercised. With contracts that have either American or partial exercise features, it is common for cashflows to arrive at different times. The timing of these cashflows does not necessarily coincide with the end of the investment horizon. Thus, the need for dynamic utility choice becomes apparent.

Allowing for early exercise translates into solving the optimal stopping problem. Pure optimal stopping problems have become a necessary component of the arbitrage-free valuation theory. A classical treatment in discrete time can be found in [51]. For a more applied recent study see also [6], who consider applications to real options in discrete time. In continuous time [1], studies the optimal stopping problem for a general jump diffusion process for a perpetual American option. Also in continuous time diffusion setting [55] studies the optimal stopping problem under the unhedgeable event risk affecting the contract's payoff. In our setting, in addition to holding the option, we allow the agent to trade in the market, which mathematically leads to a mixed optimal stopping and utility maximization problem. In complete markets, the mixed problems of optimal stopping and utility maximization have been studied by [29], who focused on the existence of optimal strategies in continuous time. In the case of an incomplete market, [7], treats the problem in the presence of transaction costs in discrete time, [9] do so continuous time. Recently, in continuous time, [14] (with a power utility) and [20] (with a static exponential utility) consider mixed optimal stopping and stochastic control problem with the non-traded assets. When a non-traded risk factor affects the contract's valuation, the models assumed in the literature are usually either simplified, so that the dependence on the non-traded factor only enters through the contract's payoff, or are very general semi-martingale models that do not allow for very explicit characterizations. Our discrete time model makes a contribution because it has the complexity to make visible the effect of the stochastic risk factor explicitly changing the dynamics of the traded asset, and at the same time provides explicit results.

A common example of a contract with partial exercise and a non-traded risk factor would be employee stock option grants. An employee may be restricted in trading his own company's stock, but may trade another correlated asset. In this case the company's stock can be viewed as a non-traded risk factor and the employee's stock options viewed as options on the non-traded risk factor.

When the market is complete and the options are written on the traded asset, the timing of early exercise is independent of the number of options held. Moreover, all the options held should optimally be exercised at once. Starting with [47], it has been pointed out that in an incomplete market, it may be optimal to exercise only a fractional amount of the total number of options held. [47] introduce the market incompleteness through the no-short-sales constraint, which causes the options to be exercised in fractions. [33] study a particular case of employee stock options, the reload options, again under the no-short-sales of the company's stock. Neither of the above publications assume the agent can trade an asset correlated to the company's stock. Allowing the executive to take a position in a correlated asset has become the next step in the development of a methodology for valuation of employee stock options. The correlated asset available for trade is typically assumed to be represented by a market index. This modeling approach was undertaken by [22] and lately by [34]. Both of these papers use the indifference valuation framework with a classical static exponential utility function and demonstrate important insights into how the risk aversion and vesting affect the value of employee stock options. In both of the above papers the authors choose to work with a simplified model where the payoff of the contract is the only part affected by the non-traded risk factor. In discrete time, [18] adopts the setting of [41] and builds on their results for European contracts, to value employee stock options. The setting of [41], as will be shown in this work, uses a reduced model, where the traded and the non-traded components are only related through the contract's payoff. In addition, the utility function used in [41] and in [18] is the traditional static exponential utility.

We present our main results in a discrete time and discrete space binomial framework, that extends the results of [41], and consequently those of [18]. We construct the backward and the forward dynamic utility processes, as [42] in continuous time. We discuss the properties of the dynamic utilities constructed, and present the algorithm that recursively computes indifference values for both dynamic utilities. We demonstrate that in both cases the prices are obtained through the recursive application of the nonlinear pricing functionals, similar to the ones introduced by [41]. We demonstrate that the prices satisfy the natural properties of being monotone with respect to risk aversion, and investigate the limiting cases as risk aversion becomes zero or infinity. We pay attention to the choice of equivalent martingale measures used for valuation with the forward and backward utilities. We provide explicit characterization of the two measures, the minimal martingale measure and the minimal entropy measure, by looking at the conditional probabilities they assign to a non-traded risk factor, given the value of the traded asset. [49] and [35] characterize the minimal martingale measure in discrete time using martingale representation properties. Our characterization of the minimal martingale measure may be less general, but is more intuitive, and we show that it satisfies the condition of [49] as well. [16] characterizes the minimal entropy measure in a discrete time discrete state model slightly different from ours. The author does not make explicit assumptions about the presence of the non-traded risk factor. He suggests that the density of the minimal entropy measure is of the exponential form, and that the minimal entropy measure assigns larger probabilities to extreme values of the traded asset. Again, our model assumptions are different and our characterization comes from a different perspective. We characterize and compare both measures and provide conditions for them to coincide. Results of chapter 3 discussing the relation between the measures have also been presented in [53].

On the pricing side, we provide an explicit formula that relates the backward and the forward prices for European contracts, and the conditions under which the two prices coincide. Since the market is incomplete, the operators involved in pricing are nonlinear with respect to the contract's payoff. For European claims we investigate the additive and multiplicative properties of those nonlinear operators with respect to the payoffs. We compare our discrete time results for European contracts to the ones of [42] in terms of the structural form of the prices, the measures used, and the characterization of the aggregate minimal entropy, and find many similarities. Chapter 4 of this work discusses European contracts, and is mostly formed of results presented in [38] and [39]. For American claims we derive the pricing algorithm, the optimal stopping policy and the hedging strategy. We also provide an alternative characterization of the American price as the supremum of European prices. With forward utility, we extend our discrete time framework to contracts with partial exercise. We also extend the continuous time results of [42] to the case of American claims, and compare the discrete and continuous American prices as well. We provide numerical examples when the backward and the forward prices are different in discrete time. We illustrate our American algorithm with a discrete time numerical example that shows the position of the free boundary. We also show a discrete time numerical example that demonstrates that the optimal exercise may indeed be partial.

As our analysis shows, with the forward utility, the prices do not depend on the end of the investment horizon. The measure used in forward valuation is completely horizon independent as well. We value American contracts of finite maturity and impose a condition that at maturity the contracts must be exercised if they have not been exercise before. This is the only reference to any kind of future time horizon. This indicates the forward utility may be a promising concept for valuing perpetual American contracts. The problem of valuating perpetual American options using utility maximization remains a question of interest in the literature. One of the difficulties is that most of the utility functions or processes used so far carry a reference to some future end of the investment horizon. Thus it has not been clear how that dependence would reflect in the valuation of perpetual contracts. Recently [21] have formulated a concept of so-called horizon-unbiased utility functions with the prospect of valuating infinite horizon real options, which is similar to the forward utility function we use. There, the authors again work with a reduced model. They suggest to adjusting the traditional static power type utility function by a multiplicative time-dependent exponential factor to make the utility dynamic. The authors also show that their dynamic utility satisfies the property of time-consistency, which is similar to the concept of self-generation developed in [42], and is the one we use in our work. Compared to [21], the forward dynamic utility suggested in [42] and developed herein is formulated in a richer, non-reduced model. In our work, instead of imposing a particular form of the utility, the latter is naturally constructed from the self-generation property and the normalization condition. As a result, the functional form of the utility is financially intuitive: the local entropy terms used to describe the forward and the backward dynamic utilities are fundamental model quantities that reflect the market incompleteness. In this work we provide the valuation algorithm and pricing examples with the forward utility, extending the scope of applications of this new concept.

Chapter 2

The discrete time model and existing results

2.1 The Model

In our model, financial instruments available for trade include a risky asset S_t and a riskless bond. Interest rate, for simplicity, is assumed to be zero. Incompleteness is introduced in the form of a non-traded risky stochastic factor Y_t. Our model adopts the setting of [41] with the two risky assets satisfying, respectively, for $0 \leq t \leq T-1$, such that:

$$\begin{cases} \dfrac{S_{t+1}}{S_t} = \xi_{t+1}, \quad \text{with} \quad \xi_{t+1} = \xi^u_{t+1}, \xi^d_{t+1}, \quad 0 < \xi^d_{t+1} < 1 < \xi^u_{t+1}, \\ \dfrac{Y_{t+1}}{Y_t} = \eta_{t+1}, \quad \text{with} \quad \eta_{t+1} = \eta^u_{t+1}, \eta^d_{t+1}, \quad 0 < \eta^d_{t+1} < \eta^u_{t+1}, \end{cases} \tag{2.1}$$

Time T represents the end of the investment horizon. The two-dimensional stochastic process $\{S_t, Y_t,\ t = 0, 1, ..., T\}$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathcal{F}_t$ generated by $\{S_s, Y_s,\ s = 1, ..., t\}$ and $\mathcal{F} = \mathcal{F}_T$. $\mathcal{F}^S_t$ denotes the filtration generated by $\{S_s,\ s = 1, ..., t\}$. $\mathbb{P}$ represents the historical measure. In general, ξ_{t+1} and η_{t+1} are allowed to depend on $S_0, ..., S_t$ and $Y_0, ..., Y_t$, but neither on S_{t+1} or Y_{t+1}.

We consider a single investor with initial wealth $X_0 = x$. He allocates wealth between the traded asset S and the risk-free bond (with zero interest, for simplicity). The initial capital and the self-financing trading strategy $\alpha = (\alpha_1, ..., \alpha_t)$ generate the time t wealth of the investor of the form:

$$X_t = x + \sum_{i=1}^{T} \alpha_i (S_i - S_{i-1}), \tag{2.2}$$

where α_s stands for the number of shares of the traded security the investor holds over the period $[s-1, s)$. We take α_s to be $\mathcal{F}_{s-1}$ measurable so that anticipation of future stock movements is not allowed.

The investor is offered an opportunity to purchase a contract with an early exercise or, in later chapters, partial exercise feature. Intrinsic payoff of the contract is specified as $C_t = C(S_t, Y_t)$ for a bounded function C. The maturity of the contract must not coincide with the end of the trading horizon. The former will be denoted by $\bar{T}$, assuming $\bar{T} \leq T$. A stopping time τ with respect to the filtration $\mathcal{F}_t$, $0 \leq t \leq \bar{T}$, represents the exercise time chosen by the investor, and is assumed to satisfy a.s. $\tau \leq \bar{T}$. Listed below are common simplifying assumptions used in the literature to value contracts that depend on a non-traded risky stochastic factor in an incomplete market with a utility framework. We will be referring to these assumptions in this work, gradually eliminating them.

Assumption 2.1 (Simplified model) *ξ_t^u and ξ_t^d, $t = 0, ..., T$ are a set of constants, and the historical distribution of the traded asset is independent of previous and current values of the non-traded risk factor, namely:*

$$\mathbb{P}(\xi_{t+1}/\mathcal{F}_t) = \mathbb{P}(\xi_{t+1}/\mathcal{F}_t^S), \; t = 0, 1, ..., T. \tag{2.3}$$

Assumption 2.2 (European claim) *The contract pays amount $C_{\bar{T}} = C(S_{\bar{T}}, Y_{\bar{T}})$ at time $\bar{T}$.*

Assumption 2.3 (Static exponential preferences) *The agent has static exponential preferences, i.e., utility function of the form $U(x) = -e^{-\gamma x}$, fixed at T that coincides with $\bar{T}$.*

We start by providing the necessary definitions for stating existing results, which have been derived under assumptions listed above for a simplified model, with the static exponential preferences and European contracts. Note that we use T as maturity of the contract because under assumption 2.3, $\bar{T} = T$.

Definition 2.1 *The value function of the buyer holding a claim C_T, $V^C(x,t)$ is the maximal expected utility of his time T wealth, conditioned on the currently available information. That is,*

$$V^C(x,t) = \sup_{\alpha_{t+1},\ldots,\alpha_T} E\left[-e^{-\gamma(X_T+C_T)}/\mathcal{F}_t\right]. \tag{2.4}$$

If $C_T = 0$ a.s. then (2.4) becomes the so-called *plain investment value function* for an investor who just optimizes his expected terminal wealth with no derivative contracts involved. The corresponding value function will be denoted by $V^0(x,t)$ and is equal to:

$$V^0(X_t,t) = \sup_{\alpha_{t+1},\ldots,\alpha_T} E\left[-e^{-\gamma X_T}/\mathcal{F}_t\right]. \tag{2.5}$$

Definition 2.2 *The time t buyer's indifference price of the European claim C_T is defined as the amount ν_t for which the two value functions V^C and V^0 coincide. Namely, ν_t is the amount which satisfies:*

$$V^C(X_t - \nu_t(C_T), Y_t, t) \;=\; V^0(X_t,t). \tag{2.6}$$

In a similar fashion, one could define the value function of the writer. Since underwriting the European liability C_T is equivalent to buying the European claim $-C_T$, the writer's value function equals the right hand side of definition (2.4), with C_T replaced by $-C_T$. By analogy, the writer's indifference price is defined as the amount that, if added to the writer's time t wealth, makes his value function as good as V^0. It's not difficult to see that, as those definitions imply, the buyer's and writer's indifference prices are related by the formula:

$$\tilde{\nu}_t(C_T) = -\nu_t(-C_T), \tag{2.7}$$

with $\tilde{\nu}_t(C_T)$ denoting the writer's indifference price for the liability C_T.

[41] established results for the writer's indifference price process of a European derivative in a discrete time setting of a simplified model. Relation (2.7) allows us to reformulate their results from the buyer's perspective, as necessary for our work. The next section explains the above mentioned results.

2.2 Existing results for European claims under simplifying assumptions

In this section we define a family of nonlinear operators to be used to characterize the buyer's indifference price. Equations (2.8) and (2.9) contain the necessary definitions. Then, Theorem 1 presents the main result of [41] for European claims, which lays the foundation for this study.

Definition 2.3 *Let Z_s, $0 \leq s \leq T$ be an $\mathcal{F}_s$-adapted process and Q be a martingale measure. For $t \leq s$ define the iterative pricing operator $\mathcal{E}_Q^{t,s}(Z_s)$ as follows:*

$$\begin{cases} \mathcal{E}_Q^{t,s}(Z_s) = \mathcal{E}_Q^{t,s-1}\left(\mathcal{E}_Q^{s-1,s}(Z_s)\right), \ t \leq s-1 \\ \mathcal{E}_Q^{s,s}(Z_s) = Z_s, \end{cases} \tag{2.8}$$

where

$$\mathcal{E}_Q^{s-1,s}(Z_s) = -E_Q\left[\frac{1}{\gamma}\ln E_Q\left[e^{-\gamma Z_s}/\mathcal{F}_{s-1} \vee \mathcal{F}_s^S\right]/\mathcal{F}_{s-1}\right]. \tag{2.9}$$

Theorem 1 *Let $\mathbb{Q}$ be a martingale measure satisfying, for $t = 0, 1, ..., T$,*

$$\mathbb{Q}\left(\eta_{t+1}/\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right) = \mathbb{P}\left(\eta_{t+1}/\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right). \tag{2.10}$$

(i) The indifference price $\nu_t(C_T)$ satisfies

$$\begin{cases} \nu_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{t,t+1}(e_{t+1}(C_T)), \ t < T \\ \nu_T(C_T) = C_T, \end{cases} \tag{2.11}$$

with $\mathcal{E}_{\mathbb{Q}}^{t,t+1}$ defined in equation (2.9) for $Q = \mathbb{Q}$.
(ii) The indifference price process is given by

$$e_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{t,T}(C_T), \tag{2.12}$$

with $\mathcal{E}_{\mathbb{Q}}^{t,T}$ defined in equations (2.8) and (2.9) for $Q = \mathbb{Q}$.
(iii) The pricing algorithm is consistent across time in that, for $0 \leq t \leq s \leq T$, the semigroup property

$$\nu_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{t,s}\left(\mathcal{E}_{\mathbb{Q}}^{s,T}(C_T)\right) = \mathcal{E}_{\mathbb{Q}}^{t,s}(e_s(C_T)) = \nu_t\left(\mathcal{E}_{\mathbb{Q}}^{s,T}(C_T)\right) \tag{2.13}$$

holds.

(iv) A martingale measure $\mathbb{Q}$ has property (2.10) if and only if it has the minimal relative to $\mathbb{P}$ entropy.

(v) At any time $0 < t < T$, the value function of the buyer is of the form

$$V^C(X_t, Y_t, t) = -e^{-\gamma(X_t + \nu_t(C_T)) - \mathcal{H}^{me}_{t,T}}, \tag{2.14}$$

where $\mathcal{H}^{me}_{t,T}$ is the minimal, conditional on $\mathcal{F}_t$ entropy commonly defined as

$$\mathcal{H}^{me}_{t,T} = \min_{Q \in \mathcal{Q}^e} E_Q \left[\ln \frac{Q(\cdot/\mathcal{F}_t)}{\mathbb{P}(\cdot/\mathcal{F}_t)} / \mathcal{F}_t \right], \tag{2.15}$$

where $\mathcal{Q}^e$ denotes the set of all martingale measures equivalent to $\mathbb{P}$, $Q(\cdot/\mathcal{F}_t)$ and $\mathbb{P}(\cdot/\mathcal{F}_t)$ denote restrictions of Q and $\mathbb{P}$ on $\mathcal{F}_t$.

(vi) The minimal entropy $\mathcal{H}^{me}_{t,T}$ has an explicit representation as:

$$\mathcal{H}^{me}_{t,T} = E_{\mathbb{Q}} \left[\sum_{k=t}^{T-1} h_k / \mathcal{F}_t \right], \tag{2.16}$$

where

$$h_k = q_{k+1} \ln \frac{q_{k+1}}{\mathbb{P}(\xi^u_{k+1}/\mathcal{F}_k)} + (1 - q_{k+1}) \ln \frac{(1 - q_{k+1})}{1 - \mathbb{P}(\xi^u_{k+1}/\mathcal{F}_k)}, \tag{2.17}$$

are referred to as local entropy terms and

$$q_k = \frac{1 - \xi^d_k}{\xi^u_k - \xi^d_k}. \tag{2.18}$$

Again, we would like to caution the reader that the above theorem has been derived under the assumption of simplified model; the characterization of the minimal entropy measure(2.9), as the one that preserves the conditional distribution of the non-traded risk factor (given the next period's value of the traded stock) does not hold if Assumption (2.1) of the simplified model is violated. In the next chapter we drop Assumption (2.1) and present a new characterization of the minimal entropy martingale measure. We also show which equivalent martingale measure has property (2.9) of preserving the conditional distribution of the non-traded risk factor, given the next period's value of the traded asset. Also, the result above shows that the minimal entropy, conditional on the available to the current time information , accumulates linearly by taking the expectation under the minimal entropy measure of the sum of the local entropy terms. As we show in the next chapter, in a more general model the entropy accumulates in a nonlinear way.

Chapter 3

The minimal martingale and the minimal entropy measures in the discrete model

Let $\mathcal{Q}$ be the set of martingale measures on $\mathcal{F}$ and consider, for $t = 0, 1, ..., T$, the quantities

$$H_T^{mm}\left(Q\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = E_{\mathbb{P}}\left(-\ln \frac{Q\left(\cdot\,|\mathcal{F}_t\right)}{\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)}\,|\mathcal{F}_t\right)$$

and

$$H_T^{me}\left(Q\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = E_Q\left(\ln \frac{Q\left(\cdot\,|\mathcal{F}_t\right)}{\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)}\,|\mathcal{F}_t\right),$$

where $Q \in \mathcal{Q}$ and $Q\left(\cdot\,|\mathcal{F}_t\right)$ and $\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)$ denote the restrictions of Q and $\mathbb{P}$ on $\mathcal{F}_t$.

The *minimal martingale measure* $\mathbb{Q}^{mm}\left(\cdot\,|\mathcal{F}_t\right)$ is defined as the *minimizer* of H_T^{mm}, i.e.,

$$\mathcal{H}_T^{mm}\left(\mathbb{Q}^{mm}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = \min_{Q \in \mathcal{Q}} H_T^{mm}\left(Q\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right). \tag{3.1}$$

Respectively, the minimal entropy measure $\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right)$ is the minimizer of H_T^{me}, i.e.,

$$\mathcal{H}_T^{me}\left(\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = \min_{Q \in \mathcal{Q}} H_T^{me}\left(Q\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right). \tag{3.2}$$

Most of the analysis below will involve the latter entropy. To simplify the presentation we will be using the condensed notation

$$\mathcal{H}^{me}_{t,T} = \mathcal{H}^{me}_{T}\left(\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) \tag{3.3}$$

and the terminology *aggregate entropy.*

The next several results provide explicit characterization of the corresponding measures. In continuous time both of these measures have been characterized in a number of different ways. In the discrete time, very few results for the two measures are available. The most closely related characterizations are in the form of predictable martingale representations, such as [49] and [35]. To the best of our knowledge, the characterization of the minimal entropy martingale measure presented below is new. It is discussed in more detail in [53]. Our characterizations for both measures are very explicit.

We introduce, for $t = 0, 1, ..., T$, the sets

$$A_t = \{\omega : \xi_t(\omega) = \xi^u_t\} \quad \text{and} \quad B_t = \{\omega : \eta_t(\omega) = \eta^u_t\} \tag{3.4}$$

Note that for all $Q, Q' \in \mathcal{Q}$,

$$Q\left(A_t\,|\mathcal{F}_{t-1}\right) = Q'\left(A_t\,|\mathcal{F}_{t-1}\right) = \frac{1-\xi^d_t}{\xi^u_t - \xi^d_t} \doteq q_t. \tag{3.5}$$

We recall the process h_t, which will be referred to as the *local entropy* process from here on. Let h_t, $t = 1, ...$ be given by

$$h_t \doteq q_t \ln \frac{q_t}{\mathbb{P}\left(A_t\,|\mathcal{F}_{t-1}\right)} + (1-q_t)\ln\frac{1-q_t}{1-\mathbb{P}\left(A_t\,|\mathcal{F}_{t-1}\right)} \tag{3.6}$$

where A_t is defined in (3.4), $\mathbb{P}$ is the historical probability measure and $\mathcal{F}_t$ the filtration generated by the random variables S_i and Y_i, for $i = 0, 1, ..., t$.

Lemma 2 *For $t = 1, ..., T$, $h_t \in \mathcal{F}_{t-1}$. Moreover, for all $Q \in \mathcal{Q}$,*

$$h_t = Q\left(A_t\,|\mathcal{F}_{t-1}\right)\ln\frac{Q\left(A_t\,|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t\,|\mathcal{F}_{t-1}\right)} + \left(1 - Q\left(A_t\,|\mathcal{F}_{t-1}\right)\right)\ln\frac{1-Q\left(A_t\,|\mathcal{F}_{t-1}\right)}{1-\mathbb{P}\left(A_t\,|\mathcal{F}_{t-1}\right)}. \tag{3.7}$$

The next proposition provides the characterization of the minimal martingale measure. This result has already appeared in [41]. There, the authors work under the assumption (2.1) that the historical probability distribution of the traded asset, given the information available up to the current time, does not depend on the path of the stochastic factor, and that the values of ξ_t and η_t are constant and do not depend on values of either traded stock or the non-traded stochastic factor. The authors characterize the minimal entropy measure using the condition presented below. Later we will show that under the assumption of a so-called $r$educed model the two measures coincide. The model of [41] satisfies the reduced model condition, implying that the characterization of the minimal entropy measure using the condition 3.8 is indeed appropriate. In the non-reduced model though, condition below characterizes the minimal martingale measure, and not the minimal entropy measure.

Proposition 3 *The minimal martingale measure* $\mathbb{Q}^{mm}\left(\cdot\right|\mathcal{F}_t)$ *satisfies, for* $t = 1, ..., T$,

$$\mathbb{Q}^{mm}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)=\mathbb{P}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right). \tag{3.8}$$

Proof. We need to show that for $t = 1, ..., T$,

$$\frac{\mathbb{Q}^{mm}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{mm}\left(A_t|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)},\ \frac{\mathbb{Q}^{mm}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{mm}\left(A_t|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}$$

$$\frac{\mathbb{Q}^{mm}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{mm}\left(A_t^c|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\,\mathcal{F}_{t-1}\right)},\ \frac{\mathbb{Q}^{mm}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{mm}\left(A_t^c|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\,\mathcal{F}_{t-1}\right)}.$$

Since the rest of the proof follows by similar arguments, we only show, the first equality, namely,

$$\frac{\mathbb{Q}^{mm}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{mm}\left(A_t|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}, \tag{3.9}$$

where sets A_t, B_t are defined as in (3.4). We use induction. At $t = T$,

$$E_{\mathbb{P}}\left(-\ln\frac{Q\left(\xi_T,\eta_T\,|\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(\xi_T,\eta_T\,|\mathcal{F}_{T-1}\right)}\,|\mathcal{F}_{T-1}\right)=-\mathbb{P}\left(A_TB_T|\,\mathcal{F}_{T-1}\right)\ln\frac{Q\left(A_TB_T|\,\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(A_TB_T|\,\mathcal{F}_{T-1}\right)}$$

$$-\mathbb{P}\left(A_T^cB_T|\,\mathcal{F}_{T-1}\right)\ln\frac{Q\left(A_T^cB_T|\,\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(A_T^cB_T|\,\mathcal{F}_{T-1}\right)}$$

$$-\mathbb{P}\left(A_TB_T^c|\,\mathcal{F}_{T-1}\right)\ln\frac{Q\left(A_T|\,\mathcal{F}_{T-1}\right)-Q\left(A_TB_T|\,\mathcal{F}_{T-1}\right)}{\mathbb{P}\left(A_TB_T^c|\,\mathcal{F}_{T-1}\right)}$$

$$-\mathbb{P}\left(A_T^c B_T^c \right| \mathcal{F}_{T-1}) \ln \frac{Q\left(A_T^c \right| \mathcal{F}_{T-1}) - Q\left(A_T^c B_T \right| \mathcal{F}_{T-1})}{\mathbb{P}\left(A_T^c B_T^c \middle| \mathcal{F}_{T-1}\right)},$$

and direct differentiation yields the claimed equality. Next, we assume that (3.9) holds for $t+1$ and show its validity at t. We have

$$E_{\mathbb{P}}\left(-\ln \frac{Q\left(\cdot \right| \mathcal{F}_t)}{\mathbb{P}\left(\cdot \right| \mathcal{F}_t)} \middle| \mathcal{F}_t \right)$$

$$= -E_{\mathbb{P}}\left(\ln \left(\prod_{i=t+1}^{T-1} \frac{Q\left(\xi_{i+1}\eta_{i+1} \middle| \mathcal{F}_i \right)}{\mathbb{P}\left(\xi_{i+1}\eta_{i+1} \middle| \mathcal{F}_i \right)} \right) \middle| \mathcal{F}_t \right) - E_{\mathbb{P}}\left(\ln \frac{Q\left(\xi_{t+1}\eta_{t+1} \middle| \mathcal{F}_t \right)}{\mathbb{P}\left(\xi_{t+1}\eta_{t+1} \middle| \mathcal{F}_t \right)} \middle| \mathcal{F}_t \right).$$

Because we use the single-period arguments used to establish (3.9) for $t = T$ and because the second term above depends only on $Q\left(\xi_{t+1}\eta_{t+1} \middle| \mathcal{F}_t \right)$, we easily conclude. ■

As we have mentioned before, the distribution of the next period's value of the traded asses, conditional on the information available up to the current time, is the same among all equivalent to $\mathbb{P}$ martingale measures. What differs among them is how the non-traded stochastic factor is distributed, given the known next period's value of the traded asset. The minimal martingale measure is characterized by a very natural and very simple assumption that the non-traded risk factor, given the known next period's value of the traded asset, has the same distribution as under the historical measure $\mathbb{P}$. The minimal martingale measure has been studied by [50] in context of continuous time in the context of quadratic hedging, and [54] have done so in the context of indifference pricing. [49] provides an explicit characterization of the minimal martingale measure in discrete time (denoted as $\hat{P}$) , using the Doob-Mayer decomposition of the traded stock process as follows:

$$\frac{d\hat{P}}{d\mathbb{P}} = \prod_{t=1}^{T} \frac{1 - \tilde{\lambda}_t \Delta S_t}{1 - \tilde{\lambda}_t \Delta A_t}, \tag{3.10}$$

where $\tilde{\lambda_t} = \dfrac{\Delta A_t}{E_{\mathbb{P}}[(\Delta S_t)^2/\mathcal{F}_{t-1}]}$ and $\Delta A_t = E_{\mathbb{P}}[\Delta S_t/\mathcal{F}_{t-1}]$. The process A_t with $A_0 = 0$ and $A_t - A_{t-1} = \Delta A_t$ defined above is the compensator in the unique Doob-Mayer decomposition of S_t, i.e. $S_t - S_0 - A_t$ is the martingale part of S_t.

More recently, [35], in a discrete time model similar to ours, show that the minimal martingale measure exists, is unique and is characterized by the density

$$\frac{d\hat{P}}{d\mathbb{P}} = \prod_{t=1}^{T}(1+\Delta L_t), \tag{3.11}$$

where the process $L_t, 0 \leq t \leq T$ has a predictable representation with respect to the martingale part $\tilde{M}_t$ of the discounted traded asset process $\tilde{S}_t$ of the following form:

$$L_t = 1 + \sum_{k=1}^{n} \alpha_k \Delta \tilde{M}_k, \tag{3.12}$$

and

$$\begin{aligned} \alpha_1 &= \frac{(1+r_1)^2 E_{\mathbb{P}}[\tilde{S}_1]}{S_0^2 Var\,(R_1)},\ R_1 = \frac{S_1}{S_0} - 1 \\ \alpha_n &= \frac{(1+r_n)(r_n - E_{\mathbb{P}}\,[R_n])}{S_{n-1} Var\,(R_n)},\ R_n = \frac{S_n}{S_{n-1}} - 1. \end{aligned} \tag{3.13}$$

In the next proposition we show that the measure $\mathbb{Q}^{mm}$ characterized by (3.8) does have the Radon-Nikodym derivative of the form specified above by [49]. The characterizations of [49] and of [35] are both somewhat technical, while the characterization derived in this work seems to be very simple, explicit, and intuitive.

Proposition 3.1 *The measure $\mathbb{Q}^{mm}$ characterized by (3.8), solves equation (3.10) shown above.*

Proof.

$$\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}} = \prod_{i=1}^{T} \frac{\mathbb{Q}^{mm}\left(\xi_i\eta_i \middle| \mathcal{F}_{i-1}\right)}{\mathbb{P}\left(\xi_i\eta_i \middle| \mathcal{F}_{i-1}\right)}. \tag{3.14}$$

Under condition (3.8) the above simplifies to

$$\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}} = \prod_{i=1}^{T} \frac{\mathbb{Q}^{mm}\left(\xi_i \middle| \mathcal{F}_{i-1}\right)}{\mathbb{P}\left(\xi_i \middle| \mathcal{F}_{i-1}\right)}. \tag{3.15}$$

$$\frac{\mathbb{Q}^{mm}\left(\xi_i \middle| \mathcal{F}_{i-1}\right)}{\mathbb{P}\left(\xi_i \middle| \mathcal{F}_{i-1}\right)} = \begin{cases} \dfrac{1-\xi_i^d}{\mathbb{P}\left(\xi_i = \xi_i^u \middle| \mathcal{F}_{i-1}\right)\left(\xi_i^u - \xi_i^d\right)}, & \text{for } \xi_i = \xi_i^u, \\ \dfrac{\xi_i^u - 1}{\left(1-\mathbb{P}\left(\xi_i = \xi_i^u \middle| \mathcal{F}_{i-1}\right)\right)\left(\xi_i^u - \xi_i^d\right)}, & \text{for } \xi_i = \xi_i^d. \end{cases} \tag{3.16}$$

Thus it only remains to show that $\frac{1-\tilde{\lambda}_t\Delta S_t}{1-\tilde{\lambda}_t\Delta A_t}$ equals the right side of (3.16). Direct calculation yields that

$$\frac{1-\tilde{\lambda}_t\Delta S_t}{1-\tilde{\lambda}_t\Delta A_t} == \frac{E_{\mathbb{P}}[(\Delta S_t)^2/\mathcal{F}_{t-1}]-\tilde{\lambda}_t\Delta S_t}{E_{\mathbb{P}}[(\Delta S_t)^2/\mathcal{F}_{t-1}]-\tilde{\lambda}_t\Delta A_t} \tag{3.17}$$

Also calculating directly,

$$E_{\mathbb{P}}[(\Delta S_t)^2/\mathcal{F}_{t-1}]-\tilde{\lambda}_t\Delta A_t = p(1-p)(\xi_t^u-\xi_t^d)^2, \tag{3.18}$$

$$E_{\mathbb{P}}[(\Delta S_t)^2/\mathcal{F}_{t-1}]-\tilde{\lambda}_t\Delta S_t = (1-p)(\xi_t^d-\xi_t^u)(\xi_t^d-1), \text{ for}\{w : S_t = S_t^u\}, \tag{3.19}$$

and

$$E_{\mathbb{P}}[(\Delta S_t)^2/\mathcal{F}_{t-1}]-\tilde{\lambda}_t\Delta S_t = p(\xi_t^u-\xi_t^d)(\xi_t^u-1), \text{ for}\{w : S_t = S_t^d\}, \tag{3.20}$$

where $p = \mathbb{P}(\xi_t = \xi_t^u/\mathcal{F}_{t-1})$, and the proof follows. ■

Next we provide an explicit representation of the minimal entropy measure. However, the characterization of the minimal entropy measure turns out not to be as much intuitive as the one for the minimal martingale measure. a number of authors have studied the minimal entropy measure in continuous time, including [17] using duality methods, [3] using PDE methods.

In the discrete time, [16] studied the characterization of the minimal entropy measure in a finite-state traded asset process model. Some of their results were obtained for a single-period model; others for a multi-period model. In their setup, the model is only implicitly incomplete, and no explicit assumptions about the presence or the form of the non-traded factor dynamics is made. Their most explicit minimal entropy measure characterization is given for one risk-free asset and one risky asset taking n different values in a one-period model. They characterize the minimal entropy measure in terms of the probabilities it assigns to the risky asset. They state that under the minimal martingale measure, the risky asset process is more likely to take extreme values (i.e. the smallest or the largest) rather then the values in the middle. We characterize the minimal martingale measure and the minimal entropy measure from a different perspective, by looking at the probabilities they assign to a non-traded factor. As mentioned earlier, in our model all the equivalent martingale measures assign the same probabilities to the values of the traded asset and the latter does not carry any specific information about either minimal entropy or minimal martingale measures. However, like [16], we do find that in one period the minimal entropy measure and the minimal martingale measure concide.

The characterization below uses the aggregate entropy $\mathcal{H}_{t,T}^{me}$, which was defined above in 3.2. An explicit form of the aggregate entropy is an independent result and will be shown later. Therefore, we may assume that the explicit form of the entropy is known to us and we can characterize the conditional minimal entropy probabilities in terms of the known historical probabilities and the aggregate entropy. To the best of our knowledge, the characterization below is a new result. It is discussed in more detail in [53].

Proposition 4 *The minimal entropy measure satisfies, for $t = 1, ..., T$,*

$$\frac{\mathbb{Q}^{me}\left(A_t B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(A_t \middle| \mathcal{F}_{t-1}\right)} = \frac{\dfrac{\mathbb{P}\left(A_t B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{uu}}}{\dfrac{\mathbb{P}\left(A_t B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{uu}} + \dfrac{\mathbb{P}\left(A_t B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{ud}}}, \tag{3.21}$$

$$\frac{\mathbb{Q}^{me}\left(A_t B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(A_t \middle| \mathcal{F}_{t-1}\right)} = \frac{\dfrac{\mathbb{P}\left(A_t B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{ud}}}{\dfrac{\mathbb{P}\left(A_t B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{uu}} + \dfrac{\mathbb{P}\left(A_t B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{ud}}}, \tag{3.22}$$

$$\frac{\mathbb{Q}^{me}\left(A_t^c B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} = \frac{\dfrac{\mathbb{P}\left(A_t^c B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{du}}}{\dfrac{\mathbb{P}\left(A_t^c B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{du}} + \dfrac{\mathbb{P}\left(A_t^c B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{dd}}}, \tag{3.23}$$

$$\frac{\mathbb{Q}^{me}\left(A_t^c B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} = \frac{\dfrac{\mathbb{P}\left(A_t^c B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{dd}}}{\dfrac{\mathbb{P}\left(A_t^c B_t \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{du}} + \dfrac{\mathbb{P}\left(A_t^c B_t^c \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)} e^{-\mathcal{H}_{t,T}^{dd}}}, \tag{3.24}$$

where the events A_t, A_t^c, B_t, B_t^c are as in (3.4) and $\mathcal{H}_{t,T}^{uu}$, $\mathcal{H}_{t,T}^{ud}$, $\mathcal{H}_{t,T}^{du}$, and $\mathcal{H}_{t,T}^{dd}$ are the values of the $\mathcal{F}_t-$measurable random variable $\mathcal{H}_{t,T}^{me}$, (defined in (3.2)), condi-

tional on $\mathcal{F}_{t-1}$.

Proof. We only show that

$$\frac{\mathbb{Q}^{me}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{Q}^{me}\left(\left.A_t\right|\mathcal{F}_{t-1}\right)}=\frac{\mathbb{P}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}^{uu}_{t,T}}}{\mathbb{P}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}^{uu}_{t,T}}+\mathbb{P}\left(\left.A_tB^c_t\right|\mathcal{F}_{t-1}\right)e^{-\mathcal{H}^{ud}_{t,T}}} \tag{3.25}$$

since the rest of equalities can be proved along similar arguments. We recall the definition of the minimal entropy measure (3.2) and observe that

$$E_Q\left(\ln\frac{Q\left(\cdot\left|\mathcal{F}_{t-1}\right.\right)}{\mathbb{P}\left(\cdot\left|\mathcal{F}_{t-1}\right.\right)}\left|\mathcal{F}_{t-1}\right.\right)=E_Q\left(\ln\frac{Q\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)}{\mathbb{P}\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)}\left|\mathcal{F}_{t-1}\right.\right)$$

$$+E_Q\left(E_Q\left(\ln\left(\prod_{i=t+1}^{T}\frac{Q\left(\xi_i,\eta_i\left|\mathcal{F}_{i-1}\right.\right)}{\mathbb{P}\left(\xi_i,\eta_i\left|\mathcal{F}_{i-1}\right.\right)}\right)\right|\mathcal{F}_t\right)\left|\mathcal{F}_{t-1}\right).$$

Therefore, one needs to minimize the first term over $Q\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)$ and the second term in two steps: first minimize the nested conditional expectation over $Q\left(\xi_i,\eta_i\left|\mathcal{F}_{i-1}\right.\right),\ i=t+1,...,T$ and, in turn, over $Q\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)$. Therefore it suffices to calculate

$$\min_{Q\left(\xi_t\eta_t\left|\mathcal{F}_{t-1}\right.\right)}\left(E_Q\left(\ln\frac{Q\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)}{\mathbb{P}\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)}\left|\mathcal{F}_{t-1}\right.\right)+E_Q\left(\left.\mathcal{H}^{me}_{t,T}\right|\mathcal{F}_{t-1}\right)\right)$$

where we used (3.2). Expanding yields

$$E_Q\left(\ln\frac{Q\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)}{\mathbb{P}\left(\xi_t,\eta_t\left|\mathcal{F}_{t-1}\right.\right)}\left|\mathcal{F}_{t-1}\right.\right)+E_Q\left(\left.\mathcal{H}^{me}_{t,T}\right|\mathcal{F}_{t-1}\right)$$

$$=Q\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\ln\frac{Q\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)}+Q\left(\left.A^c_tB_t\right|\mathcal{F}_{T-1}\right)\ln\frac{Q\left(\left.A^c_tB_t\right|\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left.A^c_tB^c_t\right|\mathcal{F}_{t-1}\right)}$$

$$+\left(Q\left(\left.A_t\right|\mathcal{F}_{t-1}\right)-Q\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\right)\ln\frac{\left(Q\left(\left.A_t\right|\mathcal{F}_{t-1}\right)-Q\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\right)}{\mathbb{P}\left(\left.A_tB^c_t\right|\mathcal{F}_{t-1}\right)}$$

$$+\left(Q\left(\left.A^c_t\right|\mathcal{F}_{t-1}\right)-Q\left(\left.A^c_tB_t\right|\mathcal{F}_{t-1}\right)\right)\ln\frac{\left(Q\left(\left.A^c_t\right|\mathcal{F}_{t-1}\right)-Q\left(\left.A^c_tB_t\right|\mathcal{F}_{t-1}\right)\right)}{\mathbb{P}\left(\left.A^c_tB^c_t\right|\mathcal{F}_{t-1}\right)}$$

$$+Q\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\mathcal{H}^{uu}_{t,T}+\left(Q\left(\left.A_t\right|\mathcal{F}_{t-1}\right)-Q\left(\left.A_tB_t\right|\mathcal{F}_{t-1}\right)\right)\mathcal{H}^{ud}_{t,T}$$

$$+Q\left(\left.A^d_tB_t\right|\mathcal{F}_{t-1}\right)\mathcal{H}^{du}_{t,T}+\left(Q\left(\left.A^c_t\right|\mathcal{F}_{t-1}\right)-Q\left(\left.A^c_tB_t\right|\mathcal{F}_{t-1}\right)\right)\mathcal{H}^{dd}_{t,T}.$$

Differentiating with respect to $Q\left(\left. A_tB_t\right| \mathcal{F}_{t-1}\right)$ and rearranging terms yields at the optimum

$$\ln\frac{\mathbb{Q}^{me}\left(\left. A_tB_t\right| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left. A_tB_t\right| \mathcal{F}_{t-1}\right)}-\ln\frac{\mathbb{Q}^{me}\left(\left. A_t\right| \mathcal{F}_{t-1}\right)-\mathbb{Q}^{me}\left(\left. A_tB_t\right| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(\left. A_tB_t^c\right| \mathcal{F}_{t-1}\right)}$$

$$+\mathcal{H}_{t,T}^{uu}-\mathcal{H}_{t,T}^{ud}=0$$

and we conclude. ■

Corollary 5 *The measures $\mathbb{Q}^{mm},\mathbb{Q}^{me}$ coincide at $T-1$,*

$$\mathbb{Q}^{mm}\left(\left. A_TB_T\right| \mathcal{F}_{T-1}\right)=\mathbb{Q}^{me}\left(\left. A_TB_T\right| \mathcal{F}_{T-1}\right) \tag{3.26}$$

with similar equalities for the events $A_TB_T,A_T^cB_T$ and $A_TB_T^c$.

The corollary implies that to see the difference between the minimal entropy and minimal martingale measure, one would need to consider a model of several periods. Then, for $t<T-1$ the differences emerge as formulae (3.21) and (3.8) indicate. The same result has been obtained by [16] in a single-period finite state traded asset process model. At the end of this chapter we expand on the question of when the two measures are the same.

Definition 1 *Let Z be a random variable on $(\Omega,\mathcal{F},\mathbb{P})$. For $t=0,1,...,T$, $s=t+1,...,T$ and $Q\in\mathcal{Q}$ define the nonlinear functional*

$$\left\{\begin{array}{l} \mathcal{J}_Q^{(t,t+1)}(Z)=E_Q\left(\ln E_Q\ \left(e^Z\left|\mathcal{F}_t\vee\mathcal{F}_{t+1}^S\right.\right)|\mathcal{F}_t\right), \\ \mathcal{J}_Q^{(t,s)}(Z)=\mathcal{J}_Q^{(t,t+1)}\left(\mathcal{J}_Q^{(t+1,t+2)}\left(...\mathcal{J}_Q^{(s-1,s)}(Z)\right)\right)\end{array}\right. \tag{3.27}$$

where $\mathcal{F}_t$ and $\mathcal{F}_t^S$ are the filtrations generated, respectively, by (S_i,Y_i) and S_i for $i=1,...,t$.

As one can see, the operator $\mathcal{J}_Q^{(t,t+1)}(\cdot)$ is the same as the operator $\mathcal{E}_Q^{(t,t+1)}(\cdot))$ for $\gamma=-1$. Operators $\mathcal{J}_Q^{(t,t+1)}(\cdot))$ will be used next to characterize the aggregate entropy. We emphasize that $\mathcal{J}_Q^{(t,t+1)}(\cdot))$ are independent of the risk aversion level γ. Also, one could easily see that

$$\mathcal{J}_Q^{(t,t+1)}(Z)=-\gamma\mathcal{E}_Q^{(t,t+1)}(-\frac{1}{\gamma}Z),\ t<T. \tag{3.28}$$

Next we proceed with the characterization of the aggregate minimal entropy. The latter has been characterized in, for example, [16] in a multi-period finite-state traded-asset process model. Their characterization suggests that the minimal relative entropy is the supremum over certainty equivalents of a certain class of payoffs, having non-positive expectations under the minimal entropy measure. This relates the model-specific quantity of the entropy to expectations of a somewhat artificial class of payoffs that have no connection to the model chosen. Our characterization of the entropy does not involve any additional artificial payoffs and is independent of the risk aversion and of any characteristics of the contract to be to be valued.

Proposition 6 *The aggregate minimal entropy is given by the iterative scheme*

$$\mathcal{H}_{T,T}^{me} = 0 \quad \text{and} \quad \mathcal{H}_{T-1,T}^{me} = h_T, \tag{3.29}$$

and

$$\mathcal{H}_{t,T}^{me} = h_{t+1} - \mathcal{J}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(-\mathcal{H}_{t+1,T}^{me}\right), \quad t = 0, 1, ..., T-2, \tag{3.30}$$

with $\mathcal{J}^{(t,t+1)}$ and h defined in (3.27) and (3.6), respectively. Moreover,

$$\mathcal{H}_{t,T}^{me} = -\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}\left(-\sum_{i=t+1}^{T} h_i\right)$$

The above proposition shows that the aggregate entropy under the minimal martingale measure accumulates in a non-linear way, using nonlinear functionals, very similar to the ones that are used for pricing European contracts. I signifies that the nonlinear functionals $\mathcal{J}_Q^{(t,t+1)}(\cdot)$ and $\mathcal{E}_Q^{(t,t+1)}(\cdot)$ are universal and used not only for pricing but also to express fundamental model quantities like the aggregate entropy. The above formula has a direct analog in the continuous time setting, which we consider in later chapters, and which draws a lot of similarity between the local entropy terms and the Sharpe ratio of the traded asset in continuous time.

Proof. The first equality in (3.29) is immediate while the second part follows easily from the definitions of $\mathcal{H}_{T-1,T}^{me}$ and h_T and equalities (3.9) and (3.26). We, next, establish (3.34) for $t = T-2$, i.e., that

$$\mathcal{H}_{T-2,T}^{me} = h_{T-1} - E_{\mathbb{Q}^{me}}\left(\ln E_{\mathbb{Q}^{me}}\left(e^{-\mathcal{H}_{T-1,T}^{me}}\left|\mathcal{F}_{T-2} \vee \mathcal{F}_{T-1}^{S}\right.\right)|\mathcal{F}_{T-2}\right). \tag{3.31}$$

We have

$$\mathcal{H}_{T-2,T}^{me} = E_{\mathbb{Q}^{me}}\left(\ln\frac{\mathbb{Q}^{me}\left(\xi_{T-1},\eta_{T-1}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(\xi_{T-1},\eta_{T-1}\left|\mathcal{F}_{T-2}\right.\right)}\left|\mathcal{F}_{T-2}\right.\right)$$

$$+E_{\mathbb{Q}^{me}}\left(E_{\mathbb{Q}^{me}}\left(\ln\frac{\mathbb{Q}^{me}\left(\xi_{T},\eta_{T}\right|\mathcal{F}_{T-1})}{\mathbb{P}\left(\xi_{T},\eta_{T}\left|\mathcal{F}_{T-1}\right.\right)}\right|\mathcal{F}_{T-1}\right)\left|\mathcal{F}_{T-2}\right.\right)$$

$$= E_{\mathbb{Q}^{me}}\left(\ln\frac{\mathbb{Q}^{me}\left(\xi_{T-1},\eta_{T-1}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(\xi_{T-1},\eta_{T-1}\left|\mathcal{F}_{T-2}\right.\right)}\right|\mathcal{F}_{T-2}\right) + E_{\mathbb{Q}^{me}}\left(h_T\left|\mathcal{F}_{T-2}\right.\right),$$

where we used (3.29). Next, we introduce the random variables

$$Z_{T-2}^{u} = \frac{\mathbb{P}\left(A_{T-1}B_{T-1}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}\left|\mathcal{F}_{T-2}\right.\right)}e^{-\mathcal{H}_{t-1,T}^{me,uu}} + \frac{\mathbb{P}\left(A_{T-1}B_{T-1}^{c}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}\left|\mathcal{F}_{T-2}\right.\right)}e^{-\mathcal{H}_{t-1,T}^{me,ud}}$$

and

$$Z_{T-2}^{d} = \frac{\mathbb{P}\left(A_{T-1}^{c}B_{T-1}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}^{c}\left|\mathcal{F}_{T-2}\right.\right)}e^{-\mathcal{H}_{t-1,T}^{me,du}} + \frac{\mathbb{P}\left(A_{T-1}^{c}B_{T-1}^{c}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}^{c}\left|\mathcal{F}_{T-2}\right.\right)}e^{-\mathcal{H}_{t-1,T}^{me,dd}}$$

where $\mathcal{H}_{t-1,T}^{me,uu}, \mathcal{H}_{t-1,T}^{me,ud}, \mathcal{H}_{t-1,T}^{me,du}$ and $\mathcal{H}_{t-1,T}^{me,dd}$ are the values of $\mathcal{H}_{t-1,T}^{me} \in \mathcal{F}_{t-1}$ conditional on $\mathcal{F}_{t-2}$.

Expanding the above formula for $\mathcal{H}_{T-2,T}^{me}$ gives:

$$\mathcal{H}_{T-2,T}^{me} = \mathbb{Q}^{me}\left(A_{T-1}B_{T-1}\right|\mathcal{F}_{T-2})\ln\left(\frac{\mathbb{Q}^{me}\left(A_{T-1}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}\right|\mathcal{F}_{T-2})}\frac{e^{-\mathcal{H}_{t-1,T}^{me,uu}}}{Z_{T-2}^{u}}\right)$$

$$+\mathbb{Q}^{me}\left(A_{T-1}B_{T-1}^{c}\right|\mathcal{F}_{T-2})\ln\left(\frac{\mathbb{Q}^{me}\left(A_{T-1}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}\right|\mathcal{F}_{T-2})}\frac{e^{-\mathcal{H}_{t-1,T}^{me,ud}}}{Z_{T-2}^{u}}\right)$$

$$+\mathbb{Q}^{me}\left(A_{T-1}^{c}B_{T-1}\right|\mathcal{F}_{T-2})\ln\left(\frac{\mathbb{Q}^{me}\left(A_{T-1}^{c}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}^{c}\right|\mathcal{F}_{T-2})}\frac{e^{-\mathcal{H}_{t-1,T}^{me,du}}}{Z_{T-2}^{d}}\right)$$

$$+\mathbb{Q}^{me}\left(A_{T-1}^{c}B_{T-1}^{c}\right|\mathcal{F}_{T-2})\ln\left(\frac{\mathbb{Q}^{me}\left(A_{T-1}^{c}\right|\mathcal{F}_{T-2})}{\mathbb{P}\left(A_{T-1}^{c}\right|\mathcal{F}_{T-2})}\frac{e^{-\mathcal{H}_{t-1,T}^{me,dd}}}{Z_{T-2}^{d}}\right) + E_{\mathbb{Q}^{me}}\left(h_T\right|\mathcal{F}_{T-2}).$$

Further simplification and rearrangement of terms yield

$$\mathcal{H}_{T-2,T}^{me} = -\mathcal{H}_{t-1,T}^{me,uu}\mathbb{Q}^{me}\left(A_{T-1}B_{T-1}\right|\mathcal{F}_{T-2}) - \mathcal{H}_{t-1,T}^{me,ud}\mathbb{Q}^{me}\left(A_{T-1}B_{T-1}^{c}\right|\mathcal{F}_{T-2})$$

$$
\begin{aligned}
&-\mathcal{H}^{me,du}_{t-1,T}\mathbb{Q}^{me}\left(A^c_{T-1}B_{T-1}\middle|\,\mathcal{F}_{T-2}\right)-\mathcal{H}^{me,dd}_{t-1,T}\mathbb{Q}^{me}\left(A^c_{T-1}B^c_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\\
&+\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\left(\ln\frac{\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}{\mathbb{P}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}-\ln Z^u_{T-2}\right)\\
&+\mathbb{Q}^{me}\left(A^c_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\left(\ln\frac{\mathbb{Q}^{me}\left(A^c_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}{\mathbb{P}\left(A^c_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}-\ln Z^d_{T-2}\right)+E_{\mathbb{Q}^{me}}\left(h_T\middle|\,\mathcal{F}_{T-2}\right)\\
&=\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\ln\frac{\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}{\mathbb{P}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}\\
&+\left(1-\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\right)\ln\frac{1-\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}{1-\mathbb{P}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)}\\
&-\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\ln Z^u_{T-2}-\mathbb{Q}^{me}\left(A^c_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\ln Z^d_{T-2}.
\end{aligned}
$$

Using (3.6) we obtain

$$
\mathcal{H}^{me}_{T-2,T}=h_{T-1}-\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\ln Z^u_{T-2}-\mathbb{Q}^{me}\left(A^c_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\ln Z^d_{T-2}. \quad (3.32)
$$

Observe, however, that because of (3.8),

$$
Z^u_{T-2}=E_{\mathbb{Q}^{mm}}\left(e^{-\mathcal{H}^{me}_{T-1,T}}\middle|\,\mathcal{F}_{T-2}\vee A_{T-1}\right)\quad\text{and}\quad Z^d_{T-2}=E_{\mathbb{Q}^{mm}}\left(e^{-\mathcal{H}^{me}_{T-1,T}}\middle|\,\mathcal{F}_{T-2}\vee A^c_{T-1}\right).
$$

The above, together with (3.5), yield

$$
\begin{aligned}
&\mathbb{Q}^{me}\left(A_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\ln Z^u_{T-2}+\mathbb{Q}^{me}\left(A^c_{T-1}\middle|\,\mathcal{F}_{T-2}\right)\ln Z^d_{T-2}\\
&=E_{\mathbb{Q}^{mm}}\left(\ln E_{\mathbb{Q}^{mm}}\left(e^{-h_T}\middle|\,\mathcal{F}_{T-2}\vee\mathcal{F}^S_{T-1}\right)\middle|\,\mathcal{F}_{T-2}\right).
\end{aligned}
$$

Combining the above and using $h_{T-1}\in\mathcal{F}_{T-2}$ we obtain (3.21).

To establish (3.21) for $t<T-2$ we work along similar arguments which have been omitted. ■

The next result highlights an important connection between $J^{(t,t+1)}_{\mathbb{Q}^{mm}}$ and $J^{(t,t+1)}_{\mathbb{Q}^{me}}$. It plays an important role in relating the backward and the forward prices developed in subsequent chapters. To our knowledge the result is new.

Proposition 7 *Let Z be an $\mathcal{F}_{t+1}-$measurable random variable and* $\mathcal{H}^{me}_{t+1,T},\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}$

as in (3.30) and (3.27), respectively. Then

$$\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}(Z) = \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}}\left(Z + \mathcal{H}^{me}_{t+1,T}\right) + \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(-\mathcal{H}^{me}_{t+1,T}\right).$$

Proof. By the definition of $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}$ we have

$$\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}(Z)$$

$$= \mathbb{Q}^{mm}\left(A_{t+1} \middle| \mathcal{F}_t\right) \ln\left(\frac{\mathbb{Q}^{mm}\left(A_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{Q}^{mm}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{uu}} + \frac{\mathbb{Q}^{mm}\left(A_{t+1}B^c_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{Q}^{mm}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{ud}}\right)$$

$$+\mathbb{Q}^{mm}\left(A^c_{t+1} \middle| \mathcal{F}_t\right) \ln\left(\frac{\mathbb{Q}^{mm}\left(A^c_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{Q}^{mm}\left(A^c_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{du}} + \frac{\mathbb{Q}^{mm}\left(A^c_{t+1}B^c_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{Q}^{mm}\left(A^c_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{dd}}\right)$$

$$= \mathbb{Q}^{me}\left(A_{t+1} \middle| \mathcal{F}_t\right) \ln\left(\frac{\mathbb{P}\left(A_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{uu}} + \frac{\mathbb{P}\left(A_{t+1}B^c_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{ud}}\right)$$

$$+\mathbb{Q}^{me}\left(A^c_{t+1} \middle| \mathcal{F}_t\right) \ln\left(\frac{\mathbb{P}\left(A^c_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A^c_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{du}} + \frac{\mathbb{P}\left(A^c_{t+1}B^c_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A^c_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{dd}}\right)$$

where we used (3.9) and (3.5) applied to $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$. Using the above proposition (representation of the minimal entropy measure) yields

$$\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}(Z)$$

$$= \mathbb{Q}^{me}\left(A_{t+1} \middle| \mathcal{F}_t\right) \ln\left(\frac{\mathbb{Q}^{me}\left(A_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{Q}^{me}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{uu}+\mathcal{H}^{me,\mathrm{u}}_{t+1,T}} \times\right.$$

$$\times\left(\frac{\mathbb{P}\left(A_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{-\mathcal{H}^{me,\mathrm{u}}_{t+1,T}} + \frac{\mathbb{P}\left(A_{t+1}B^c_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{-\mathcal{H}^{me,ud}_{t+1,T}}\right)$$

$$+\frac{\mathbb{Q}^{me}\left(A_{t+1}B^c_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{Q}^{me}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{ud}+\mathcal{H}^{me,ud}_{t+1}} \times$$

$$\times\left(\frac{\mathbb{P}\left(A_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{-\mathcal{H}^{me,\mathrm{u}}_{t+1,T}} + \frac{\mathbb{P}\left(A_{t+1}B^c_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{P}\left(A_{t+1} \middle| \mathcal{F}_t\right)} e^{-\mathcal{H}^{me,ud}_{t+1,T}}\right)$$

$$+\mathbb{Q}^{me}\left(A^c_{t+1} \middle| \mathcal{F}_t\right) \ln\left(\frac{\mathbb{Q}^{me}\left(A^c_{t+1}B_{t+1} \middle| \mathcal{F}_t\right)}{\mathbb{Q}^{me}\left(A^c_{t+1} \middle| \mathcal{F}_t\right)} e^{Z^{du}+\mathcal{H}^{me,du}_{t+1,T}} \times\right.$$

$$\times\left(\frac{\mathbb{P}\left(\left.A_{t+1}^{c}B_{t+1}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,du}}+\frac{\mathbb{P}\left(\left.A_{t+1}^{c}B_{t+1}^{c}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,dd}}\right)$$

$$+\frac{\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}B_{t+1}^{c}\right|\mathcal{F}_{t}\right)}{\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{Z^{dd}+\mathcal{H}_{t+1,T}^{me,dd}}\times$$

$$\times\left(\frac{\mathbb{P}\left(\left.A_{t+1}^{c}B_{t+1}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,du}}+\frac{\mathbb{P}\left(\left.A_{t+1}^{c}B_{t+1}^{c}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,dd}}\right).$$

Further simplification yields

$$\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,t+1)}(Z)=\mathbb{Q}^{me}\left(\left.A_{t+1}\right|\mathcal{F}_{t}\right)\ln\left(\frac{\mathbb{Q}^{me}\left(\left.A_{t+1}B_{t+1}\right|\mathcal{F}_{t}\right)}{\mathbb{Q}^{me}\left(\left.A_{t+1}\right|\mathcal{F}_{t}\right)}e^{Z^{uu}+\mathcal{H}_{t+1,T}^{uu}}\right.$$

$$\left.+\frac{\mathbb{Q}^{me}\left(\left.A_{t+1}B_{t+1}^{c}\right|\mathcal{F}_{t}\right)}{\mathbb{Q}^{me}\left(\left.A_{t+1}\right|\mathcal{F}_{t}\right)}e^{Z^{ud}+\mathcal{H}_{t+1,T}^{me,ud}}\right)$$

$$+\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)\ln\left(\frac{\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}B_{t+1}\right|\mathcal{F}_{t}\right)}{\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{Z^{du}+\mathcal{H}_{t+1,T}^{me,du}}\right.$$

$$\left.+\frac{\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}B_{t+1}^{c}\right|\mathcal{F}_{t}\right)}{\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{Z^{dd}+\mathcal{H}_{t+1,T}^{me,dd}}\right).$$

$$+\mathbb{Q}^{me}\left(\left.A_{t+1}\right|\mathcal{F}_{t}\right)\ln\left(\frac{\mathbb{P}\left(\left.A_{t+1}B_{t+1}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,\mathrm{u}}}+\frac{\mathbb{P}\left(\left.A_{t+1}B_{t+1}^{c}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,ud}}\right)$$

$$+\mathbb{Q}^{me}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)\ln\left(\frac{\mathbb{P}\left(\left.A_{t+1}^{c}B_{t+1}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,du}}+\frac{\mathbb{P}\left(\left.A_{t+1}^{c}B_{t+1}^{c}\right|\mathcal{F}_{t}\right)}{\mathbb{P}\left(\left.A_{t+1}^{c}\right|\mathcal{F}_{t}\right)}e^{-\mathcal{H}_{t+1,T}^{me,dd}}\right)$$

and the claim follows. ■

Corollary 8 *Let* $\mathcal{H}_{t,T}^{me}$ *be the aggregate entropy. Then*

$$\mathcal{J}_{\mathbb{Q}^{me}}^{(t,t+1)}\left(\mathcal{H}_{t+1,T}^{me}\right)=-\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(-\mathcal{H}_{t+1,T}^{me}\right). \tag{3.33}$$

We next obtain the representation of the aggregate entropy $\mathcal{H}_{t,T}^{me}$ in terms of the minimal entropy measure. The proof follows from Proposition 8 and the measurability properties of h.

Proposition 9 *The aggregate minimal entropy is given by the iterative scheme*

$$\mathcal{H}^{me}_{T,T} = 0 \quad \textit{and} \quad \mathcal{H}^{me}_{T-1,T} = h_T,$$

and

$$\mathcal{H}^{me}_{t,T} = h_{t+1} + \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}} \left(\mathcal{H}^{me}_{t,T}\right), \quad t = 0, 1, ..., T-2, \tag{3.34}$$

with $\mathcal{J}^{(t,t+1)}$ and h defined in (3.27) and (3.6), respectively. Moreover,

$$\mathcal{H}^{me}_{t,T} = \mathcal{J}^{(t,T)}_{\mathbb{Q}^{me}} \left(\sum_{i=t+1}^{T} h_i \right). \tag{3.35}$$

A natural question one should ask is when the minimal entropy measure is the same as the minimal martingale measure. This question has been investigated by several authors in continuous time, including [25] in the context of a stochastic volatility model. They concluded that the two measures coincide when the Sharpe ratio is either constant or $\mathcal{F}^S$ - measurable (i.e., a function of the traded asset only, and not of its non-traded volatility). [45] call this latter case a complete model. In a more general model, [56] provide sufficient conditions for $\mathbb{Q}^{mm} = \mathbb{Q}^{me}$, also in continuous time, using predictable representations with respect to the martingale part of the traded asset process. The condition we derived in our discrete time model is simple and is similar to the Sharpe ratio condition of [45]. It implies that the local entropy terms are $\mathcal{F}_S$-measurable (i.e., they do not depend on the non-traded risk factor). The condition of the next corollary impacts considerably the representation of the minimal entropy, of the value function and the indifference prices that will be developed in later chapters. Throughout this work, the condition below will be referred to as the *reduced model* in discrete time.

Corollary 10 (Reduced model) *If the historical distribution of the traded asset is independent of the past and current values of the non-traded risk factor, namely*

$$\mathbb{P}(\xi_{t+1}/\mathcal{F}_t) = \mathbb{P}(\xi_{t+1}/\mathcal{F}^S_t), \ t = 0, 1, ..., T, \tag{3.36}$$

and neither are ξ^u_{t+1} and ξ^d_{t+1} (i.e., ξ^u_{t+1} and ξ^d_{t+1} are $\mathcal{F}^S_t$-measurable), then
(i) the local entropy terms h_u are $\mathcal{F}^S_u$-measurable, $u = 0, 1, ..., T$

(ii) the representation of the aggregate entropy $\mathcal{H}_{t,T}^{me}$ *simplifies to*

$$\mathcal{H}_{t,T}^{me} = E_{\mathbb{Q}^{mm}}\left[\sum_{i=t+1}^{T} h_i/\mathcal{F}_t\right], \tag{3.37}$$

(ii) the minimal martingale measure $\mathbb{Q}^{mm}$ *coincides with the minimal entropy measure* $\mathbb{Q}^{me}$*.*

Note that $\mathcal{F}_t^S$-measurability of ξ_{t+1}^u and ξ_{t+1}^d *does not* make S_{t+1} $\mathcal{F}_t^S$-measurable.

Proof. (i) follows easily from the definition of the local entropy terms h_u (equation (3.6)). (ii) As easily seen from the explicit formula for $\mathcal{J}_{\mathbb{Q}^{me}}$ (equation (3.27)) and from the aggregate entropy representation above (equation (3.35)), under the already established condition (i),

$$\mathcal{J}_{\mathbb{Q}^{me}}^{(t,T)}(h_u) = E_{\mathbb{Q}_{me}}[h_u/\mathcal{F}_t], \text{ u=0,1,...,T}, \tag{3.38}$$

and the first part of the corollary is obvious. Since h_u is $\mathcal{F}_u$-measurable, so is $\mathcal{H}_{t,T}^{me}$. With the last property, the quantities $\mathcal{H}_{t,T}^{uu}$ and $\mathcal{H}_{t,T}^{ud}$ in equation (3.21) are the same and this equation (3.21) becomes:

$$\frac{\mathbb{Q}^{me}(A_tB_t/\mathcal{F}_{t-1})}{\mathbb{Q}^{me}(A_t/\mathcal{F}_{t-1})} = \frac{\mathbb{P}(A_tB_t/\mathcal{F}_{t-1})}{\mathbb{P}(A_t/\mathcal{F}_{t-1})}, \tag{3.39}$$

with the corresponding equations for other combinations of sets A, A^c, B and B^c. Using the characterization (3.8) of the minimal martingale measure, it become obvious that the measures coincide. ■

We finish with a representation result for the value function. This is the classical value function, with the classical static exponential utility, studied in continuous time, among others, by [46], and [10], [28] and [2]. In [13] have obtained a representation of V^0 in discrete time using duality methods for a general static strictly increasing, concave, continuous, and continuously differentiable utility function, fixed at the terminal time horizon. Their results were derived in a model with M traded assets and L non-tradable risk factors, but with only one trading period, which starts at $t = 0$ and ends at $t = 1$. Because it is a one-period model, the latter has the minimal martingale measure equal to the minimal entropy measure. The result presented below is, in the above mentioned sense, more general than that of [13], and represents the value function in a different form.

Proposition 11 *The value function satisfies, for $x \in \mathcal{R}$ and $t = 0, 1, ..., T$,*

$$V^0(x,t) = -\exp\left(-\gamma x - \mathcal{H}^{me}_{t,T}\right).$$

Thus,

$$\begin{aligned} V^0(x,t) &= -\exp\left(-\gamma x - \mathcal{J}^{(t,T)}_{\mathbb{Q}^{me}}\left(\sum_{i=t+1}^{T} h_i\right)\right) \\ &= -\exp\left(-\gamma x + \mathcal{J}^{(t,T)}_{\mathbb{Q}^{mm}}\left(-\sum_{i=t+1}^{T} h_i\right)\right) \end{aligned}$$

with h and $\mathcal{J}$ as in (3.6) and (3.27) respectively.

Chapter 4

Dynamic indifference valuation for European contracts in the discrete model.

4.1 Dynamic indifference valuation with the forward utility.

A common assumption in the classical utility maximization approach to valuation of derivative contracts in incomplete markets is that the investor optimizes his expected wealth, generated by the contract held and through the choice of trading strategy over a certain time period. The end of that time period is often referred to as an investment horizon. The utility function is traditionally specified at the end of the investment horizon. In this framework, even in the absence of any liabilities, the optimal expected terminal wealth depends on the end of the investment horizon. The above indicates that the classical utility maximization approach with the utility function fixed at the end of the investment horizon might create mispricing and misalignment in investment decisions. Such problems may be avoided if one chooses to work with dynamic utility processes. The dynamic utility process, as shown in [42], can be constructed to solve the inconsistency problem caused by the choice of investment horizon necessary for the model.

The choice for the dynamic exponential utility that yields the same expected wealth levels across different investment horizons is not unique. Such a utility process could propagate forward in time and be pre-specified at some initial time point (called the normalization point), thus generating a forward utility, or it could be pre-specified at a future time (again, called the normalization point) and propagate backward, generating a backward utility. In this section we present a discrete time forward utility. Following the ideas of [42], we construct the corresponding forward utility process. The process generates the same wealth levels for different investment horizons. It also satisfies the so-called self-generation property that the maximal expected wealth, generated with the given utility process and starting with the current wealth level in the absence of any derivative contacts) is the same as the utility of the current wealth level. We explain how European contracts can be priced with such a utility process. We derive the recursive pricing algorithm, and study how the prices are effected by risk aversion. We show that the prices are only sub-linear with respect to payoffs, unlike the pricing by expectation in complete market. More importantly, we emphasize that not only the forward utility process, but also the prices, are independent of the investment horizon. Even though the forward utility process is dependent on the initial normalization point, the prices generated by the algorithm are not. We believe the last two properties make the forward dynamic utility an attractive candidate for use in the valuation of derivative contracts.

We start by presenting a small lemma that is a technical intermediate result, and will used later to derive the desired properties for the forward utility. The proof of the lemma follows easily from formula (11) presented in chapter 3. We recall that (11) is the discrete time analog for the classical result of [10] which characterizes the functional form of the so-called pain investment value function in utility maximization problems with static exponential utility.

As mentioned earlier, our model is general enough to value contracts for which maturity does not necessarily coincide with the end of the investment horizon. From now on, we denote by T the end of the investment horizon and by $\bar{T}$ the contract's maturity, and assume $\bar{T} \leq T$.

Lemma 12 *For $t \geq 0$ and $i = t, t+1, ..., T$, consider the aggregate entropy $\mathcal{H}_T$ and the local entropies h_i, given, respectively, in (2.17) in Chapter 2 and recalled in*

(3.6) in Chapter 3. Then $\mathcal{H}_T \in \mathcal{F}_t$ while $h_i \in \mathcal{F}_{i-1}$. Moreover,

$$\sup_{\alpha_{i+1}} E_{\mathbb{P}} \left(-e^{-\gamma \alpha_{i+1} \Delta S_{i+1} + h_{i+1}} \middle| \mathcal{F}_i \right) = -1. \tag{4.1}$$

We are now ready to introduce the forward dynamic utility in a multi-period setting. For this, we fix the *forward normalization point* denoted by $s \geq 0$. The forward dynamic utility, denoted by $U_t^F(x; s)$ is defined for $t \geq s$ below.

Definition 4.1 *Let $s \geq 0$ be the forward normalization point. For $t = s, s+1, ...,$, an $\mathcal{F}_t$-measurable stochastic process $U_t^F(x; s)$ is called a forward dynamic exponential utility, normalized at s, if, for all t, T, with $T = t+1, t+2, ...,$ it satisfies the stochastic optimization criterion*

$$U_t^F(x; s) = \begin{cases} -e^{-\gamma x}, & t = s \\ \\ \sup\limits_{\mathcal{A}} E_{\mathbb{P}} \left(U_T^F(X_T; s) \middle| \mathcal{F}_t \right), & s \leq t \leq T \end{cases} \tag{4.2}$$

with X_T as in (2.2) and $X_t = x$.

Several things to notice are that by construction, there is no constraint on the length of the trading horizon and that the investor receives the same expected utility of terminal wealth, independently of the length of the trading horizon. The latter property will be referred to as the time consistency of the dynamic utility. Also, the forward dynamic utility is self-generating (the value function) is defined traditionally as the expected utility of the terminal wealth, and coincides with the utility function itself.

Note that the forward dynamic utility might not be unique. Questions about uniqueness and robustness are currently under investigation and are not the subject of this presentation. We will be working with one specific solution of (4.2), introduced below.

Proposition 13 *For $s = 0, 1, ...,$ the process $\left\{ U_t^F(x; s) : t = s, s+1, ... \right\}$ defined, for $x \in \mathbb{R}$, by*

$$U_t^F(x; s) = \begin{cases} -e^{-\gamma x}, & t = s \\ \\ -e^{-\gamma x + \sum_{u=s+1}^{t} h_u}, & t \geq s+1 \end{cases} \tag{4.3}$$

h-variables with the local entropy terms, h_i, $i = 1, 2, ...$, given by (3.6), is a forward dynamic exponential utility.

Proof. The fact that U^F is normalized at time s is a direct consequence of its definition. To show (4.3), it suffices to establish it for $T = t+1$ since the rest of the proof would follow by direct induction arguments. To this end, we recall

$$U_t^F(x; s) = \sup_{\alpha_{t+1}} E_{\mathbb{P}}\left(U_{t+1}^F(X_{t+1}; s) \,|\mathcal{F}_t\right) \tag{4.4}$$

$$= \sup_{\alpha_{t+1}} E_{\mathbb{P}}\left(-e^{-\gamma X_{t+1} + \Sigma_{i=s+1}^{t+1} h_i} \,|\mathcal{F}_t\right) \tag{4.5}$$

$$= e^{-\gamma x + \Sigma_{i=s+1}^{t} h_i} \sup_{\alpha_{t+1}} E_{\mathbb{P}}\left(-e^{-\gamma \alpha_{t+1} \Delta S_{t+1} + h_{t+1}} \,|\mathcal{F}_T\right) \tag{4.6}$$

and using (4.1) we conclude. ■

In this section, we discuss the notion of forward indifference price and we provide an iterative algorithm for its construction. This new concept of price was recently introduced by the authors in [42] in continuous time. In discrete time the issue has been considered in [38], and this section follows closely that paper, with the only exception that all the results are formulated from the buyer's point of view.

Definition 4.2 *Let $s \geq 0$ be the forward normalization point and consider the forward dynamic exponential utility U^F introduced in (4.3). Let $s \leq t_0 \leq \bar{T}$ and consider a claim, written at t_0, yielding payoff $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$. For $t \in [s, \bar{T}]$, the forward indifference price is defined as the amount $\nu_t^F(C_{\bar{T}}; s)$ for which*

$$U_t^F\left(x + \nu_t^F(C_{\bar{T}}; s); s\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_{\bar{T}}^F\left(X_{\bar{T}} + C_{\bar{T}}; s\right)\big|\, \mathcal{F}_t\right) \tag{4.7}$$

for all initial wealth levels $x \in \mathbb{R}$.

The notation we use suggests that there is a dependency of the forward dynamic utility and, consequently, the price on the normalization point s. As will become clear after the explicit formula for the price is derived, the normalization point does not affect the price, but for now we keep the normalization point in the notation.

Next, we construct the forward indifference valuation algorithm. In the definition of the forward price functionals, the forward dynamic utility and its inverse, with respect to the wealth argument, appear. This functional is denoted by $\left(U^F\right)_t^{-1}(x;s)$ and is given, for $t \geq s$, by

$$\left(U^F\right)_t^{-1}(x;s) = -\frac{1}{\gamma}\log(-x) + \frac{1}{\gamma}\sum_{i=s+1}^{t} h_i, \tag{4.8}$$

for $x \in \mathbb{R}^-$ and h_i as in (3.6). We recall that $\mathcal{F}_t$ and $\mathcal{F}_t^S$ are the filtrations generated, respectively, by the random variables (S_i, Y_i), and S_i, for $i = 1, 2, ..., t$.

Definition 4.3 *Let Z be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Let s be the forward normalization point and $U_t^F(x)$ and $\left(U^F\right)_t^{-1}(x)$ the forward dynamic utility and its inverse (given in (4.3) and (4.8)). We define:*

· the forward pricing measure $\mathbb{Q}^{mm}$ to be the minimal martingale measure, characterized by:

$$\mathbb{Q}^{mm}\left(\eta_{t+1}\left|\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right.\right) = \mathbb{P}\left(\eta_{t+1}\left|\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right.\right),\ t = 0, 1, ..., \bar{T}. \tag{4.9}$$

· the single step forward price functional

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(Z;s) = E_{\mathbb{Q}^{mm}}\left(\left(U^F\right)_{t+1}^{-1}\left(E_{\mathbb{Q}^{mm}}\ \left(U_{t+1}^F(Z;s)\left|\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right.\right);s\right)\right), \tag{4.10}$$

· the multi step forward price functional

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t')}(Z;s) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t+1,t+2)}(...(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t'-1,t')}(Z;s);s);s);s), \tag{4.11}$$

across the time interval (t, t').

Theorem 14 *Let $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$ be the claim, introduced at t_0 and to be priced under the forward dynamic utility $U_t^F(x;s)$. The following statements hold:*

(i) The forward indifference price $\nu_t^F(C_{\bar{T}};s)$, defined in (4.7), is given, for $t_0 \leq t \leq \bar{T}$, by the backward induction algorithm

$$\begin{cases} \nu_t^F(C_{\bar{T}};s) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(\nu_{t+1}^F(C_{\bar{T}};s);s),\ t < \bar{T}\ , \\ \nu_{\bar{T}}^F(C_{\bar{T}};s) = C_{\bar{T}}, \end{cases} \tag{4.12}$$

with $\mathbb{Q}^{mm}$ defined in (4.9).

(ii) The forward indifference price process $\nu_t^F(C_{\bar{T}}; s) \in \mathcal{F}_t$ and satisfies, for $t \geq t_0$,

$$\nu_t^F(C_{\bar{T}}; s) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,\bar{T})}(C_{\bar{T}}; s), \tag{4.13}$$

with the iterative forward price functional defined in (4.11).

(iii) The forward indifference price algorithm is consistent across time in that, for $s \leq t \leq t' \leq \bar{T}$, the semigroup property

$$\nu_t^F(C_{\bar{T}}; s) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t')}(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t',\bar{T})}(C_{\bar{T}}; s); s) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t')}(\nu_{t'}^F(C_{\bar{T}}; s); s) = \nu_t^F(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,\bar{T})}(C_{\bar{T}}; s); s). \tag{4.14}$$

holds.

Before we prove the theorem, we present a proposition below. The following result emphasizes two important properties of the single-step forward price functional, namely, its *independence of the forward normalization point* and its *static nature.* The first property is the one that mainly differentiates the forward and the backward pricing schemes. Note that the forward price functional $\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,\bar{T})}(\cdot)$ emerges from the pricing condition (4.7) in which the involved forward dynamic utility does depend on s and, also changes dynamically in time. Nevertheless, these two features dissipate in the price. In addition, the proposition re-formulates equation (4.12). This is the form in which theorem 14 will be proved.

Proposition 15 *Let Z be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(\cdot)$is the single-step forward price functional, introduced in (4.10), then for $t = s, s+1, ..., \bar{T} - 1$,*

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(Z; s) = -E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma Z}\left|\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right.\right)|\mathcal{F}_t\right). \tag{4.15}$$

Proof. We first observe that

$$E_{\mathbb{Q}^{mm}}\left(U_{t+1}^F(Z; s)\left|\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right.\right) = E_{\mathbb{Q}^{mm}}\left(-e^{-\gamma Z + \Sigma_{i=s+1}^{\bar{T}} h_i}\right|\left.\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right) \tag{4.16}$$

$$= -e^{\Sigma_{i=s+1}^{\bar{T}} h_i} E_{\mathbb{Q}^{mm}}\left(-e^{-\gamma Z}\right|\left.\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right). \tag{4.17}$$

Therefore,

$$\left(U^F\right)_{t'}^{-1}\left(E_{\mathbb{Q}^{mm}}\left(U_{t'}^F(Z; s)\left|\mathcal{F}_t \vee \mathcal{F}_{t'}^S\right.\right); s\right) \tag{4.18}$$

$$= -\frac{1}{\gamma}\log\left(e^{\Sigma_{i=s+1}^{\bar{T}}h_i} E_{\mathbb{Q}^{mm}}\left(-e^{-\gamma Z}\middle|\, \mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right)\right) + \frac{1}{\gamma}\sum_{i=s+1}^{t} h_i \tag{4.19}$$

$$= -\frac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(-e^{-\gamma Z}\middle|\, \mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right) \tag{4.20}$$

and the assertion follows. ∎

We are now ready to prove theorem 14. **Proof.** We prove (i) in the view of propostion 15 above. (ii) and (iii) follow from the definition of operators $\mathcal{E}^{t,u}_{\mathbb{Q}^{mm}}(\cdot)$, $t \le u \le \bar{T}$. At time $\bar{T}-1$,

$$\begin{gathered}\sup_{\mathcal{A}} E_{\mathbb{P}}\left(U^F_{\bar{T}}\left(X_{\bar{T}} + C_{\bar{T}}; s\right)\middle|\, \mathcal{F}_{\bar{T}-1}\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}}+C_{\bar{T}})+h_{\bar{T}}}\middle|\, \mathcal{F}_{\bar{T}-1}\right) = \\ e^{h_{\bar{T}}} \sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}}+C_{\bar{T}})}\middle|\, \mathcal{F}_{\bar{T}-1}\right) = e^{h_{\bar{T}}}\left(-e^{-\gamma(X_{\bar{T}-1}+\mathcal{E}^{(\bar{T}-1,\bar{T})}_{\mathbb{Q}^{mm}}(C_{\bar{T}}))-h_{\bar{T}}}\right),\end{gathered} \tag{4.21}$$

as follows using theorem 1, Chapter 2, Part (v).

At time $\bar{T}-2$,

$$\begin{gathered}\sup_{\mathcal{A}} E_{\mathbb{P}}\left(U^F_{\bar{T}}\left(X_{\bar{T}} + C_{\bar{T}}; s\right)\middle|\, \mathcal{F}_{\bar{T}-2}\right) = \sup_{\alpha_{\bar{T}-1},\alpha_{\bar{T}}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}}+C_{\bar{T}})+h_{\bar{T}}+h_{\bar{T}-1}}\middle|\, \mathcal{F}_{\bar{T}-2}\right) = \\ e^{h_{\bar{T}-1}} \sup_{\alpha_{\bar{T}-1}} E_{\mathbb{P}}\left(e^{h_{\bar{T}}}\sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}}+C_{\bar{T}})}\middle|\, \mathcal{F}_{\bar{T}-1}\right)|\mathcal{F}_{\bar{T}-2}\right) = \\ e^{h_{\bar{T}-1}} \sup_{\alpha_{\bar{T}-1}} E_{\mathbb{P}}\left(e^{h_{\bar{T}}} - e^{-\gamma(X_{\bar{T}-1}+\nu^F_{\bar{T}-1}(C_{\bar{T}};s))-h_{\bar{T}}}|\mathcal{F}_{\bar{T}-1}\right) = \\ -e^{-\gamma(X_{\bar{T}-2}+\mathcal{E}^{(\bar{T}-2,\bar{T}-1)}_{\mathbb{Q}^{mm}}(\nu^F_{\bar{T}-1}(C_{\bar{T}};s)))}.\end{gathered} \tag{4.22}$$

For other $t < T-2$, the argument is similar and is omitted. Taking into account proposition 15 presented above, the proof of 14 is complete. ∎

The forward indifference price is calculated via the iterative pricing scheme (4.12), applied backwards in time. The scheme has local and dynamic properties. Dynamically, at each time interval, say $[t; t+1)$, the price $\nu^F_t(C_{\bar{T}}; s)$ is computed via the single-step forward price functional $\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}(\cdot)$, applied to the claim's value at the end of period $[t; t+1)$. The latter turns out to be the forward indifference price, $\nu^F_{t+1}(C_{\bar{T}}; s)$, yielding forward prices consistent across times. Locally, the valuation role of $\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}$ is similar to its static counterpart, as developed in [41], it is *nonlinear* and produces the prices in *two sub-steps.* In the first sub-step, the end of the period value, $\nu^F_{t+1}(C_{\bar{T}}; s)$, is altered using forward risk preferences and the conditioning on the information generated by $\mathcal{F}_t \vee \mathcal{F}^S_{t+1}$. The new payoff, called the *forward conditional certainty equivalent*

$$\tilde{\nu}^F_{t+1}(C_{\bar{T}}; s) = \left(U^F\right)^{-1}_{t+1} \left(E_{\mathbb{Q}^{mm}} \left(U^F_{t+1} \left(\nu^F_{t+1}(C_{\bar{T}}; s)\right) | \mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right); s\right) \tag{4.23}$$

$$= -\frac{1}{\gamma} \log E_{\mathbb{Q}^{mm}} \left(e^{-\gamma \nu^F_{t+1}(C_{\bar{T}}; s)} | \mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right) \tag{4.24}$$

emerges. Once this step is executed, the remaining risks are priced linearly. The forward indifference price is then provided as

$$\nu^F_t(C_{\bar{T}}; s) = E_{\mathbb{Q}^{mm}}(\tilde{\nu}^F_{t+1}(C_{\bar{T}}; s) | \mathcal{F}_t). \tag{4.25}$$

The price functional $\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}(\cdot)$ is affected by the forward dynamic risk preferences (and their inverse) only at its first sub-step. Note, however, that $\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}(\cdot)$ is independent of the specific payoff and uses the same pricing measure throughout. It is also independent of the forward normalization point, a property that is, in turn, inherited by the forward prices. To remind ourselves of this, we therefore eliminate the $s-$notation.

The reader familiar with existing indifference algorithmic results for multi-period binomial models (see, among others, [52], [2] and [41]) might find the form of (4.12) very similar to the one appearing in these references. However, the results in the latter works and the ones herein differ considerably. First, the pricing measure used is neither the minimal entropy measure, nor the historical measure, but the minimal martingale measure. Second, earlier results refer to entirely different risk preference structures and are derived for simplified market conditions, where the next period's distribution of the traded asset is not affected by the value of the non-traded stochastic factor.

The algorithm presented above for the indifference valuation with the forward dynamic utility may be extended to contracts that yield a series of clash flows. This is not a subject of this work. Instead, in later chapters, we develop the corresponding algorithm for early exercise and partial exercise claims.

We study the behavior of the forward indifference prices in terms of the model parameters. We start by presenting how the indifference is affected by changes in risk aversion . We occasionally adopt the notation $\nu^F_t(C_{\bar{T}}; \gamma)$ and $\mathcal{E}^{(t,\bar{T})}_{\mathbb{Q}^{mm}}(\cdot\ ; \gamma)$ to refer to a specific level of risk aversion.

Proposition 16 *For a fixed contract $C_{\bar{T}}$, the function*

$$\gamma \to \nu_t^F(C_{\bar{T}};\gamma) \tag{4.26}$$

from R_+ into R is decreasing and continuous.

Proof. Continuity follows from formula (4.12) and the continuity properties of the single-step price functional $\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(\cdot)$, for $t = s, s+1, ..., \bar{T}-1$.

To establish the claimed monotonicity, we take $\gamma_1 \le \gamma_2$ and use backward induction. We first show that

$$\nu_{\bar{T}-1}^F(C_{\bar{T}};\gamma_1) \ge \nu_{\bar{T}-1}^F(C_{\bar{T}};\gamma_2) \tag{4.27}$$

or, equivalently, that

$$-E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma_1}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_1 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right) \tag{4.28}$$

$$\ge -E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma_2}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_2 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right). \tag{4.29}$$

To this end, using Holder's inequality yields

$$E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_1 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right) \le \left(E_{\mathbb{Q}^{mm}}\ \left(e^{-\gamma_2 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right)\right)^{\gamma_1/\gamma_2} \tag{4.30}$$

and, in turn,

$$-\frac{1}{\gamma_1}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_1 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right) \ge -\frac{1}{\gamma_2}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_2 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right). \tag{4.31}$$

Taking conditional expectation with respect to $\mathcal{F}_{\bar{T}-1}$ and under $\mathbb{Q}^{mm}$, we conclude. We next assume that

$$\nu_{t+1}^F(C_{\bar{T}};\gamma_1) \ge \nu_{t+1}^F(C_{\bar{T}};\gamma_2) \tag{4.32}$$

and we are going to establish

$$\nu_t^F(C_{\bar{T}};\gamma_1) \ge \nu_t^F(C_{\bar{T}};\gamma_2). \tag{4.33}$$

Using (4.12), the monotonicity of $\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}$ with respect to the payoff and Holder's inequality, we obtain

$$\nu_t^F(C_{\bar{T}};\gamma_1) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(\nu_{t+1}^F(C_{\bar{T}};\gamma_1);\gamma_1) \geq \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(\nu_{t+1}^F(C_{\bar{T}};\gamma_2);\gamma_1) \tag{4.34}$$

$$\geq \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}(\nu_{t+1}^F(C_{\bar{T}};\gamma_2);\gamma_2) = \nu_t^F(C_{\bar{T}};\gamma_2), \tag{4.35}$$

which completes the proof. ■

Proposition 17 *The following limiting relations hold*

$$\lim_{\gamma\downarrow 0} \nu_t^F(C_{\bar{T}};\gamma) = E_{\mathbb{Q}^{mm}}\left(C_{\bar{T}}\,|\mathcal{F}_t\right) \tag{4.36}$$

and

$$\lim_{\gamma\uparrow\infty} \nu_t^F(C_{\bar{T}};\gamma) = \nu_t^{F,Inf}(C), \tag{4.37}$$

with $\nu_t^{F,Inf}(C)$ *defined recursively as:*

$$\begin{cases} \nu_{\bar{T}}^{F,Inf}(C) = C_{\bar{T}}, \\ \nu_t^{F,Inf}(C) = E_{\mathbb{Q}^{mm}}(\inf_{Y_{t+1}} \nu_{t+1}^{F,Inf}(C)),\ t < \bar{T}. \end{cases} \tag{4.38}$$

Proof. We recall that

$$\nu_t^F(C_{\bar{T}}) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t+1,t+2)}\left(...\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(\bar{T}-1,\bar{T})}\left(C_{\bar{T}}\right)\right)\right)\right) \tag{4.39}$$

with

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(i,i+1)}(Z) = -E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma Z}\left|\mathcal{F}_i \vee \mathcal{F}_{i+1}^S\right.\right)|\mathcal{F}_i\right), \tag{4.40}$$

for $i = t, t+1, ..., \bar{T}-1$ and Z a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. For convenience, we denote by t_j the generic intermediate point of $[t, \bar{T}]$, i.e. $t = t_1 \leq t_2 \leq ... \leq t_{n-1} \leq \bar{T} = t_n$, and we define, for $j = 1, ..., n-1$,

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t_j,t_{j+1})}(Z;\,\gamma_{j+1}) = -E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma_{j+1}}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_{j+1}Z}\left|\mathcal{F}_{t_j} \vee \mathcal{F}_{t_{j+1}}^S\right.\right)|\mathcal{F}_{t_j}\right). \tag{4.41}$$

The forward indifference price can then be written as

$$\nu_t^F(C_{\bar{T}};\gamma_1,...,\gamma_n) \tag{4.42}$$

$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_1,t_2)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t_2,t_3)}\left(\left(...\mathcal{E}_{\mathbb{Q}^{mm}}^{(\bar{T}-1,\bar{T})}\left(C_{\bar{T}};\gamma_n\right)\right);\gamma_2\right);\gamma_1\right) \tag{4.43}$$

with $\gamma_j = \gamma$ for $j = 1,...,n$. Using the continuity and monotonicity properties of the forward single-step price functional, we deduce that

$$\lim_{\gamma\to\bar{\gamma}} \nu_t^F(C_{\bar{T}};\gamma) = \lim_{\gamma_1\to\bar{\gamma}}\left(...\left(\lim_{\gamma_n\to\bar{\gamma}}\nu_t^F(C_{\bar{T}};\gamma_1,...,\gamma_n)\right)\right) \tag{4.44}$$

$$= \lim_{\gamma_1\to\bar{\gamma}}\mathcal{E}_{\mathbb{Q}^{mm}}^{(t_1,t_2)}\left(\lim_{\gamma_2\to\bar{\gamma}}\mathcal{E}_{\mathbb{Q}^{mm}}^{(t_2,t_3)}\left(\left(...\lim_{\gamma_n\to\bar{\gamma}}\mathcal{E}_{\mathbb{Q}^{mm}}^{(t_n,t_{n+1})}\left(C_{\bar{T}};\gamma_n\right)\right);\gamma_2\right);\gamma_1\right). \tag{4.45}$$

The limits in (4.36) and (4.37) then correspond to the choices $\bar{\gamma} = 0$ and $\bar{\gamma} = \infty$.

We start with the case $\bar{\gamma} = 0$. Clearly, the analysis reduces to the specification of the individual limits, as $\gamma_m \downarrow 0$ of the nested single forward price functionals. We observe

$$\lim_{\gamma_n\downarrow 0}\mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right) \tag{4.46}$$

$$= \lim_{\gamma_n\downarrow 0} -E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma_n}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_n C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right) \tag{4.47}$$

where we remind the reader that $t_{n-1} = \bar{T} - 1$ and $t_n = \bar{T}$. The above term expands as:

$$\begin{aligned}
&\nu_{\bar{T}-1}^F(C_{\bar{T}};\gamma_n) = \mathcal{E}_{\mathbb{Q}^{mm}}^{\bar{T}-1,\bar{T}}(C_{\bar{T}}) = -\\
&\frac{\mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\gamma_n}\log\left(\frac{\mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}e^{-\gamma_n C_{\bar{T}}^{uu}} + \frac{\mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}e^{-\gamma_n C_{\bar{T}}^{ud}}\right) -\\
&\frac{\mathbb{Q}^{mm}(A_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})}{\gamma_n}\log\left(\frac{\mathbb{Q}^{mm}(A_{\bar{T}}^cB_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})}e^{-\gamma_n C_{\bar{T}}^{du}} + \frac{\mathbb{Q}^{mm}(A_{\bar{T}}^cB_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})}e^{-\gamma_n C_{\bar{T}}^{dd}}\right),
\end{aligned} \tag{4.48}$$

where events $A_{\bar{T}}$, $A_{\bar{T}}^c$, $B_{\bar{T}}$ and $B_{\bar{T}}^c$ are defined in equation (3.4) of Chapter 3.

Sending $\gamma_n \downarrow 0$ yields

$$\lim_{\gamma_n \downarrow 0} \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right) = \mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})C_{\bar{T}}^{uu}$$
$$+\mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}^{C}/\mathcal{F}_{\bar{T}-1})C_{\bar{T}}^{ud} + \mathbb{Q}^{mm}(A_{\bar{T}}^{c}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})C_{\bar{T}}^{du} + \mathbb{Q}^{mm}(A_{\bar{T}}^{c}B_{\bar{T}}^{c}/\mathcal{F}_{\bar{T}-1})C_{\bar{T}}^{dd}. \tag{4.49}$$

Combining the above yields

$$\lim_{\gamma_n \downarrow 0} \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right) = \lim_{\gamma_n \downarrow 0} \mathcal{E}_{\mathbb{Q}^{mm}}^{(\bar{T}-1,\bar{T})}\left(C_{\bar{T}};\gamma_n\right) = E_{\mathbb{Q}^{mm}}(C_{\bar{T}}\left|\mathcal{F}_{\bar{T}-1}\right.). \tag{4.50}$$

Repeating the arguments as $\gamma_{n-1} \downarrow 0, ..., \gamma_1 \downarrow 0,$ we deduce that

$$\lim_{\gamma \downarrow 0} \nu_{\bar{T}}^{F}(C_{\bar{T}};\gamma) = E_{\mathbb{Q}^{mm}}\left(\left(...E_{\mathbb{Q}^{mm}}\left(E_{\mathbb{Q}^{mm}}(C_{\bar{T}}\left|\mathcal{F}_{\bar{T}-1}\right.)\left|\mathcal{F}_{\bar{T}-2}\right.\right)\right)...\left|\mathcal{F}_t\right.\right), \tag{4.51}$$

and using the properties of conditional expectation we obtain (4.36).

Next, we look at the case $\bar{\gamma} = \infty$ and we follow along the lines of the above arguments. We start by calculating $\lim_{\gamma \uparrow \infty} \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right)$. Using (4.48) and passing to the limit as $\gamma_n \uparrow \infty$, we observe

$$\lim_{\gamma_n \uparrow \infty} \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right) \tag{4.52}$$

$$= \lim_{\gamma_n \uparrow \infty} -E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma_n}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma_n C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1} \vee \mathcal{F}_{\bar{T}}^{S}\right.\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right) \tag{4.53}$$

$$= \mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\min\left(C_{\bar{T}}^{uu}, C_{\bar{T}}^{du}\right) + \left(1 - \mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\right)\min\left(C_{\bar{T}}^{du}, C_{\bar{T}}^{dd}\right)$$
$$= E_{\mathbb{Q}^{mm}}(\inf_{Y_{\bar{T}}} C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}) \tag{4.54}$$

Using the continuity of the involved single-step forward price functionals, we deduce

$$\lim_{\gamma_{n-1} \uparrow \infty} \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-2},t_{n-1})}\left(\lim_{\gamma_n \uparrow \infty} \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right);\gamma_{n-1}\right) \tag{4.55}$$

$$= \lim_{\gamma_{n-1} \uparrow \infty} \mathcal{E}_{\mathbb{Q}^{mm}}^{(t_{n-2},t_{n-1})}\left(E_{\mathbb{Q}^{mm}}(\inf_{Y_{\bar{T}}} C_{\bar{T}}/\mathcal{F}_{\bar{T}-1});\gamma_{n-1}\right). \tag{4.56}$$

As $\gamma_{n-1} \uparrow \infty$,

$$\nu_{t_{n-2}}^{F}(C_{\bar{T}};\gamma) \downarrow \nu_{t_{n-2}}^{F,Inf}(C). \tag{4.57}$$

Working similarly for the rest of the nested limits we obtain (4.37). ■

We note that results of the above two Propositions have been established for indifference prices under the traditional sense (backward) and for general market environments by several authors (see, among others, [46], [10]).

We explore the monotonicity, convexity, and scaling properties of the forward indifference prices. We note that in the next two propositions, all inequalities among payoffs and their prices hold both under the historical measure $\mathbb{P}$ and the pricing measure $\mathbb{Q}^{mm}$. Since these two measures are equivalent, we skip, for the ease of the presentation, any measure-specific notation.

Proposition 18 *The forward indifference price is a non-decreasing function of the claim's payoff, namely, for $t = t_0, t_0 + 1, ..., \bar{T}$,*

$$\text{if } \; C^1_{\bar{T}} \leq C^2_{\bar{T}} \quad \text{then } \; \nu^F_t \left(C^1_{\bar{T}}\right) \leq \nu^F_t \left(C^2_{\bar{T}}\right), \; . \tag{4.58}$$

In addition, if $\alpha \in (0, 1)$, and $C \geq 0$ then

$$\nu^F_t \left(\alpha C^1_{\bar{T}} + (1-\alpha) C^2_{\bar{T}}\right) \geq \alpha \nu^F_t \left(C^1_{\bar{T}}\right) + (1-\alpha)\nu^F_t \left(C^2_{\bar{T}}\right). \tag{4.59}$$

Proof. Monotonicity follows from elementary backward induction arguments, the valuation algorithm (4.12) and the monotonicity of $\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}(\cdot)$ with respect its payoff argument.

The induction arguments needed for the convexity property are more involved, and for these we present the key steps only. We first establish (4.59) for $t = \bar{T} - 1$, i.e. that

$$\nu^F_{\bar{T}-1} \left(\alpha C^1_{\bar{T}} + (1-\alpha) C^2_{\bar{T}}\right) \leq \alpha \nu^F_{\bar{T}-1} \left(C^1_{\bar{T}}\right) + (1-\alpha)\nu^F_{\bar{T}-1} \left(C^2_{\bar{T}}\right). \tag{4.60}$$

To this end, we use (4.12) and Holder's inequality to deduce

$$\nu^F_{\bar{T}-1} \left(\alpha C^1_{\bar{T}} + (1-\alpha) C^2_{\bar{T}}\right) \tag{4.61}$$

$$
\begin{aligned}
&= -E_{\mathbb{Q}^{mm}}\left(\tfrac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma\left(\alpha C^1_{\bar{T}}+(1-\alpha)C^2_{\bar{T}}\right)}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}^S_{\bar{T}}\right.\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right)\\
&\quad\geq -E_{\mathbb{Q}^{mm}}\left(\tfrac{1}{\gamma}\log\left(\left(E_{\mathbb{Q}^{mm}}\left(e^{-\gamma C^1_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}^S_{\bar{T}}\right.\right)\right)^{\alpha}\right.\right.\\
&\qquad\times\left.\left.\left(E_{\mathbb{Q}^{mm}}\left(e^{-\gamma C^2_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}^S_{\bar{T}}\right.\right)\right)^{1-\alpha}\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right) \qquad (4.62)\\
&= -\alpha E_{\mathbb{Q}^{mm}}\left(\tfrac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma C^1_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}^S_{\bar{T}}\right.\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right)\\
&-(1-\alpha)\,E_{\mathbb{Q}^{mm}}\left(\tfrac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma C^2_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}^S_{\bar{T}}\right.\right)\left|\mathcal{F}_{\bar{T}-1}\right.\right).
\end{aligned}
$$

We now assume that

$$
\nu^F_{i+1}\left(\alpha C^1_{\bar{T}}+(1-\alpha)C^2_{\bar{T}}\right)\geq\alpha\nu^F_{i+1}\left(C^1_{\bar{T}}\right)+(1-\alpha)\nu^F_{i+1}\left(C^2_{\bar{T}}\right) \tag{4.63}
$$

and we will establish

$$
\nu^F_{i}\left(\alpha C^1_{\bar{T}}+(1-\alpha)C^2_{\bar{T}}\right)\geq\alpha\nu^F_{i}\left(C^1_{\bar{T}}\right)+(1-\alpha)\nu^F\left(C^2_{\bar{T}}\right). \tag{4.64}
$$

Using (4.12) and the above assumption, we observe that

$$
\nu^F_{i}\left(\alpha C^1_{\bar{T}}+(1-\alpha)C^2_{\bar{T}}\right) \tag{4.65}
$$

$$
= E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{\gamma\nu^F_{i+1}\left(\alpha C^1_{\bar{T}}+(1-\alpha)C^2_{\bar{T}}\right)}\left|\mathcal{F}_i\vee\mathcal{F}^S_{i+1}\right.\right)|\mathcal{F}_i\right) \tag{4.66}
$$

$$
\geq E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{\gamma\left(\alpha\nu^F_{i+1}\left(C^1_{\bar{T}}\right)+(1-\alpha)\nu^F_{i+1}\left(C^2_{\bar{T}}\right)\right)}\left|\mathcal{F}_i\vee\mathcal{F}^S_{i+1}\right.\right)|\mathcal{F}_i\right). \tag{4.67}
$$

Applying Holder's inequality to the last expectation yields

$$
\nu^F_{i}\left(\alpha C^1_{\bar{T}}+(1-\alpha)C^2_{\bar{T}}\right) \tag{4.68}
$$

$$
\geq -E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\log\left(E_{\mathbb{Q}^{mm}}\left(e^{-\gamma\nu^F_{i+1}\left(C^1_{\bar{T}}\right)}\left|\mathcal{F}_i\vee\mathcal{F}^S_{i+1}\right.\right)^{\alpha}\right.\right. \tag{4.69}
$$

$$
\left.\left.\times\ E_{\mathbb{Q}^{mm}}\left(e^{-\gamma\nu^F_{i+1}\left(C^2_{\bar{T}}\right)}\left|\mathcal{F}_i\vee\mathcal{F}^S_{i+1}\right.\right)^{1-\alpha}\right)|\mathcal{F}_i\right) \tag{4.70}
$$

$$
= -\alpha E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma\nu^F_{i+1}\left(C^1_{\bar{T}}\right)}\left|\mathcal{F}_i\vee\mathcal{F}^S_{i+1}\right.\right)|\mathcal{F}_i\right) \tag{4.71}
$$

$$
-(1-\alpha)E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}^{mm}}\left(e^{-\gamma\nu^F_{i+1}\left(C^2_{\bar{T}}\right)}\left|\mathcal{F}_i\vee\mathcal{F}^S_{i+1}\right.\right)|\mathcal{F}_i\right) \tag{4.72}
$$

$$= \alpha \nu_i^F \left(C_{\bar{T}}^1 \right) + (1-\alpha) \nu_i^F \left(C_{\bar{T}}^2 \right), \tag{4.73}$$

and we easily conclude. ■

Next, we investigate additivity properties of the forward indifference price with respect to the claims' payoffs. As expected, multiples of the same payoff are not priced by forward indifference in a linear manner. This is a direct consequence of the market incompleteness and the nonlinear character of forward valuation. Also, for two different claims, the forward price fails to act additively. Nothing, however, can be further said about the emerging non-linearities, since they strongly depend on the specific model and payoff structure. A concise characterization of the family of claims that are priced additively under forward indifference valuation remains, to the best of our knowledge, an open question.

Proposition 19 *The forward indifference price satisfies, for $C \geq 0$ and $t = t_0, t_0 + 1, ..., \bar{T}$,*

$$\nu_t^F \left(\alpha C_{\bar{T}} \right) \geq \alpha \nu_t^F \left(C_{\bar{T}} \right) \quad \text{for } \alpha \in (0,1) \tag{4.74}$$

and

$$\nu_t^F \left(\alpha C_{\bar{T}} \right) \leq \alpha \nu_t^F \left(C_{\bar{T}} \right) \quad \text{for } \alpha \geq 1. \tag{4.75}$$

Proof. We only show (4.74) since (4.75) follows along similar arguments. We first establish that

$$\nu_{\bar{T}-1}^F \left(\alpha C_{\bar{T}} \right) \geq \alpha \mathcal{E}_{\mathbb{Q}^{mm}}^{(\bar{T}-1,\bar{T})} (C_{\bar{T}}). \tag{4.76}$$

Indeed, from Proposition 15 we easily see that

$$\nu_{\bar{T}-1}^F \left(\alpha C_{\bar{T}} \right) = -E_{\mathbb{Q}^{mm}} \left(\frac{1}{\gamma} \log E_{\mathbb{Q}^{mm}} \left(e^{-\gamma \alpha C_{\bar{T}}} \left| \mathcal{F}_{\bar{T}-1} \vee \mathcal{F}_{\bar{T}}^S \right) \right| \mathcal{F}_{\bar{T}-1} \right) \tag{4.77}$$

$$= -\alpha E_{\mathbb{Q}^{mm}} \left(\frac{1}{\bar{\gamma}} \log E_{\mathbb{Q}^{mm}} \left(e^{-\bar{\gamma} C_{\bar{T}}} \left| \mathcal{F}_{\bar{T}-1} \vee \mathcal{F}_{\bar{T}}^S \right) \right| \mathcal{F}_{\bar{T}-1} \right) \tag{4.78}$$

$$\geq -\alpha E_{\mathbb{Q}^{mm}} \left(\frac{1}{\gamma} \log E_{\mathbb{Q}^{mm}} \left(e^{-\gamma C_{\bar{T}}} / \mathcal{F}_{\bar{T}-1} \vee \mathcal{F}_{\bar{T}}^S \right) / \mathcal{F}_{\bar{T}-1} \right), \tag{4.79}$$

where we used monotonicity of the price with respect to the risk aversion coefficient (see Proposition 16) for $\alpha \in (0,1)$ and $\bar{\gamma} = \alpha \gamma < \gamma$. Dividing by $\alpha > 0$, yields the desired inequality.

Working with backward induction, we next assume that

$$\nu_{i+1}^{F}\left(\alpha C_{\bar{T}}\right) \geq \alpha \nu_{i+1}^{F}\left(C_{\bar{T}}\right). \tag{4.80}$$

The forward valuation algorithm, together with the induction assumption above, yield

$$\nu_{i}^{F}\left(\alpha C_{\bar{T}}\right) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(i,i+1)}(\nu_{i+1}^{F}\left(\alpha C_{\bar{T}}\right)) \tag{4.81}$$

$$\geq \mathcal{E}_{\mathbb{Q}^{mm}}^{(i,i+1)}(\alpha \nu_{i+1}^{F}\left(C_{\bar{T}}\right)). \tag{4.82}$$

Using arguments similar to the above, we obtain

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(i,i+1)}(\alpha \nu_{i+1}^{F}\left(C_{\bar{T}}\right)) \geq \alpha \mathcal{E}_{\mathbb{Q}^{mm}}^{(i,i+1)}(\nu_{i+1}^{F}\left(C_{\bar{T}}\right)) \tag{4.83}$$

and conclude the proof. ■

We finish this section by showing that the process $\nu_{t}^{F}\left(C_{\bar{T}}\right)$ is a submartingale with respect to $\mathcal{F}_t$, under the pricing measure $\mathbb{Q}^{mm}$. The proof is simple and follows theorem 14 (the main pricing result) and Jensen's inequality, as thus the proof is omitted.

Proposition 20 *The forward indifference price process is a submartingale with respect to $\mathcal{F}_t$ and under $\mathbb{Q}^{mm}$, namely, for $t = t_0, t_0 + 1, ...\bar{T}$,*

$$E_{\mathbb{Q}^{mm}}\left(\nu_{t+1}^{F}\left(C_{\bar{T}}\right)|\mathcal{F}_t\right) \geq \nu_{t}^{F}\left(C_{\bar{T}}\right). \tag{4.84}$$

4.2 Dynamic indifference valuation with the backward utility.

In this section we consider a European contract initiated at time t_0 and expires at time $\bar{T}$, yielding payoff $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$. The utility of the agent is fixed at time $T > \bar{T}$, the time T is referred to as the end of the investment horizon. Making utility fixed at a future time point is a characteristic feature of the traditional static exponential utility and backward utility developed herein as well. Again, herein we consider valuation form the buyer's prospective, so the relevant formulas from [39] are all re-stated to reflect the buyer's point of view.

Definition 4.4 *A stochastic process* $\left\{U_t^B(x;T) : t = 0, 1, ..., \bar{T}\right\}$ *is called a backward exponential utility, normalized at time* $\bar{T}$*, if it satisfies the following properties*

- *at the normalization point it coincides with the utility datum,*

$$U_{\bar{T}}^B(x;T) = -e^{-\gamma x} \tag{4.85}$$

- *for all admissible self-financing investment policies* α*, the process* $U_t^B(X_t^\alpha;T)$ *is an* $\mathcal{F}_t-$ *supermartingale under* $\mathbb{P}$*,*

$$E_{\mathbb{P}}\left(U_s^B(X_s^\alpha;T)\middle|\,\mathcal{F}_t\right) \le U_t^B(X_t^\alpha;T) \qquad s = t, t+1, ..., \bar{T} \tag{4.86}$$

- *there exists a policy,* α^**, for which the process* $U_t^B\left(X_t^{\alpha^*};T\right)$ *is a* $\mathcal{F}_t-$ *martingale under* $\mathbb{P}$*,*

$$E_{\mathbb{P}}\left(U_s^B(X_s^\alpha;T)\middle|\,\mathcal{F}_t\right) = U_t^B(X_t^\alpha;T) \qquad s = t, t+1, ..., \bar{T}. \tag{4.87}$$

The last two properties in the above definition can be rewritten as

$$U_t^B(X_t;T) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_T^B(X_T;s)\middle|\,\mathcal{F}_t\right), \qquad s \le t \le T \tag{4.88}$$

We used property (4.88) when formulating the forward utility, and as one can see, it is equivalent to (4.86) and (4.87). One might wonder if the backward utility process exists and, moreover, if it is unique. The answer is affirmative as the next result states. The proof follows directly from the Dynamic Programming Principle.

Proposition 4.1 *The exponential backward utility process is unique and is given by the "plain investment" value function* $V^0(X_t, t)$*, defined in 2.4 of Chapter 2 and recalled in proposition 11 of Chapter 3.*

$$V^0(x,t) = U_t^B(x;T) = -e^{-\gamma x - \mathcal{H}_{t,\bar{T}}^{me}} \tag{4.89}$$

where $\mathcal{H}_{t,\bar{T}}^{me}$ *is the aggregate entropy.*

Next, we define the indifference price process.

Definition 4.5 *Let $U_t^B(x;T)$, $t=0,1,...,\bar{T}$ be the exponential backward utility process given in (4.89). The indifference price of the claim $C_{\bar{T}}=C(S_{\bar{T}},Y_{\bar{T}})$, $C_{\bar{T}}\in\mathcal{F}_{\bar{T}}$, is defined as the amount $\nu_t^B(C_{\bar{T}};T)$, $t=0,1,...,\bar{T}$, for which equality*

$$U_t^B\left(X_t+\nu_t^B(C_{\bar{T}};T);T\right)=\sup_{\alpha_{t+1},...,\alpha_{\bar{T}}}E_{\mathbb{P}}\left(U_{\bar{T}}^B(X_{\bar{T}}+C_{\bar{T}};T)\middle|\,\mathcal{F}_t\right) \tag{4.90}$$

is satisfied for all initial wealth levels $x\in\mathcal{R}$.

Below we provide two alternative variations of the indifference pricing algorithm, one under the minimal martingale measure, the other one under the minimal entropy measure. First we first present the algorithm under the minimal martingale measure, and make a comment about the difference in the algorithms for backward and forward utilities. We note that expressions of the first algorithm can be simplified if one works under the minimal entropy measure instead.

Thus we provide another version of the algorithm, this one under the minimal entropy measure. We mention that the second version of the algorithm is very much in line with the continuous time indifference valuation with backward utility, as shown by [54]. Next, we investigate the properties of the prices obtained through our algorithm. At the end of the section we compare and contrast the backward and forward valuation algorithms.

The following functionals will be used in the sequel.

Definition 4.6 *Let Z be a random variable on $(\Omega,\mathcal{F},\mathbb{P})$. For $t=0,1,...,\bar{T}$, $s=t+1,...,\bar{T}$ and $Q\in\mathcal{Q}$ define*

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,t+1)}(Z)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(Z+\frac{1}{\gamma}\mathcal{H}_{t+1,\bar{T}}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(\frac{1}{\gamma}\mathcal{H}_{t+1,\bar{T}}^{me}\right) \tag{4.91}$$

and

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}(Z)=\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(...\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-1,t)}(Z)\right). \tag{4.92}$$

Theorem 21 *Let $C_{\bar{T}}\in\mathcal{F}_{\bar{T}}$ be the claim to be priced. Let $\mathbb{Q}^{mm}$ be the minimal martingale measure and $\mathcal{H}_{t,\bar{T}}^{me}$ the aggregate entropy. The following statements are true:*

(i) The indifference price $\nu_t^B(C_{\bar{T}};T)$, defined in (4.90), is given by the algorithm

$$\nu_{\bar{T}}^B(C_{\bar{T}};T)=C_{\bar{T}}, \tag{4.93}$$

$$\nu_t^B\left(C_{\bar{T}};T\right)=\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(\nu_{t+1}^B\left(C_{\bar{T}};T\right)\right), \tag{4.94}$$

where $\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,t+1)}$ *is the single-step pricing functional given in (4.91).*

(ii) The indifference price process is given by

$$\nu_t^B(C_{\bar{T}};T)=\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,\bar{T})}\left(C_{\bar{T}}\right),\quad t=0,1,...,\bar{T}, \tag{4.95}$$

with the multi-step price functional $\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,\bar{T})}$ *defined in (4.92).*

In particular,

$$\nu_t^B(C_{\bar{T}};T)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,T)}\left(C_{\bar{T}}+\frac{1}{\gamma}\sum_{i=t+2}^{T}h_i\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t,T)}\left(+\frac{1}{\gamma}\sum_{i=t+2}^{T}h_i\right),\quad t=0,1,...,\bar{T}-1. \tag{4.96}$$

(iii) The pricing algorithm is consistent across time in that for $0\leq t\leq s\leq\bar{T}$*, the semigroup property*

$$\nu_t^B\left(C_{\bar{T}};T\right)=\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,s)}(\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,\bar{T})}\left(C_{\bar{T}}\right))=\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,s)}((\nu_s^B\left(C_{\bar{T}};T\right))=\nu_t^B(\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,\bar{T})}(C_{\bar{T}};T)) \tag{4.97}$$

holds.

(iv) The value function $V^{C_{\bar{T}}}$*, defined in (2.4), can be written in the form*

$$V^{C_{\bar{T}}}\left(x,t\right)=-\exp\left(-\gamma\left(x-\nu_t^B\left(C_{\bar{T}};T\right)\right)+\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}\left(-\sum_{i=t+1}^{T}h_i\right)\right), \tag{4.98}$$

with h *given in (3.6) and* $\mathcal{J}_{\mathbb{Q}^{mm}}^{(t,\bar{T})}$ *as in (3.27) with* $Q=\mathbb{Q}^{mm}$*.*

Proof. Equality (4.93) is immediate. We establish (4.94) for $t=\bar{T}-1$. Using the single-period arguments employed in [41], and recalled in 2.14

$$V^{C_{\bar{T}}}\left(x,\bar{T}-1\right)=\sup_{\alpha_{\bar{T}}}E_{\mathbb{P}}\left(U^B\left(x+\alpha_{\bar{T}}\left(S_{\bar{T}}-S_{\bar{T}-1}\right)-C_{\bar{T}}\right)\middle|\,\mathcal{F}_{\bar{T}-1}\right) \tag{4.99}$$

$$=-e^{-\gamma\left(x-\mathcal{E}_{\mathbb{Q}^{mm}}^{(\bar{T}-1,\bar{T})}(C_{\bar{T}})\right)-h_{\bar{T}}}. \tag{4.100}$$

Using $V^0\left(x,\bar{T}-1\right)=-e^{-\gamma x-h_{\bar{T}}}$, we deduce

$$\nu^B_{\bar{T}-1}\left(C_{\bar{T}};T\right)=\mathcal{E}^{\left(\bar{T}-1,\bar{T}\right)}_{\mathbb{Q}^{mm}}\left(C_{\bar{T}}\right). \tag{4.101}$$

For $t=\bar{T}-2$, we have

$$\begin{aligned}&V^{C_{\bar{T}}}\left(x,\bar{T}-2\right)=\\ &\sup_{\alpha_{\bar{T}-1},\alpha_{\bar{T}}}E_{\mathbb{P}}\left(-e^{-\gamma\left(x+\alpha_{\bar{T}-1}\left(S_{\bar{T}-1}-S_{\bar{T}-2}\right)+\alpha_{\bar{T}}\left(S_{\bar{T}}-S_{\bar{T}-1}\right)-C_{\bar{T}}\right)}\Big|\,\mathcal{F}_{\bar{T}-2}\right)\end{aligned} \tag{4.102}$$

$$\begin{aligned}&=\sup_{\alpha_{\bar{T}-1}}E_{\mathbb{P}}\left(e^{-\gamma\left(x+\alpha_{\bar{T}-1}\left(S_{\bar{T}-1}-S_{\bar{T}-2}\right)\right)}\right.\\ &\left.\sup_{\alpha_{\bar{T}}}E_{\mathbb{P}}\left(-e^{-\gamma\left(\alpha_{\bar{T}}\left(S_{\bar{T}}-S_{\bar{T}-1}\right)-C_{\bar{T}}\right)}\Big|\,\mathcal{F}_{\bar{T}-1}\right)\Big|\,\mathcal{F}_{\bar{T}-2}\right)\end{aligned} \tag{4.103}$$

$$\begin{aligned}&=\sup_{\alpha_{\bar{T}-1}}E_{\mathbb{P}}\left(-e^{-\gamma\left(x+\alpha_{\bar{T}-1}\left(S_{\bar{T}-1}-S_{\bar{T}-2}\right)-\left(\nu^B_{\bar{T}-1}(C_{\bar{T}};T)-\frac{1}{\gamma}h_{\bar{T}}\right)\right)}\Big|\,\mathcal{F}_{\bar{T}-2}\right)\\ &=-e^{-\gamma\left(x-\mathcal{E}^{(\bar{T}-2,\bar{T}-1)}_{\mathbb{Q}^{mm}}\left(\nu^B_{\bar{T}-1}(C_{\bar{T}};T)-\frac{1}{\gamma}h_{\bar{T}}\right)\right)-h_{\bar{T}-1}}\end{aligned} \tag{4.104}$$

where we used (4.101) and (3.28). Proposition 11 in Chapter 3 together with (3.28) and the measurability properties of h yields

$$V^0\left(x,\bar{T}-2\right)=-e^{-\gamma x+\mathcal{J}^{(\bar{T}-2,,\bar{T})}_{\mathbb{Q}^{mm}}\left(-\left(h_{\bar{T}-1}+h_{\bar{T}}\right)\right)}=-e^{-\gamma x-h_{\bar{T}-1}+\mathcal{E}^{(\bar{T}-2,,\bar{T})}_{\mathbb{Q}^{mm}}\left(-\frac{h_{\bar{T}}}{\gamma}\right)}. \tag{4.105}$$

Combining the above we obtain

$$\nu^B_{\bar{T}-2}\left(C_{\bar{T}};T\right)=\mathcal{E}^{\left(\bar{T}-2,\bar{T}-1\right)}_{\mathbb{Q}^{mm}}\left(\nu^B_{\bar{T}-1}\left(C_{\bar{T}};T\right)-\frac{1}{\gamma}h_{\bar{T}}\right)-\mathcal{E}^{\left(\bar{T}-2,\bar{T}-1\right)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}h_{\bar{T}}\right) \tag{4.106}$$

$$=\mathcal{E}^{\left(\bar{T}-2,\bar{T}-1\right)}_{\mathbb{Q}^{mm}}\left(\nu_{\bar{T}-1}\left(C_{\bar{T}}\right)-\frac{1}{\gamma}\mathcal{H}^{me}_{\bar{T}-1,\bar{T}}\right)-\mathcal{E}^{\left(\bar{T}-2,\bar{T}-1\right)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{\bar{T}-1,\bar{T}}\right). \tag{4.107}$$

Next, we assume that

$$\nu^B_{t+1}\left(C_{\bar{T}};T\right)=\mathcal{E}^{(t+1,t+2)}_{\mathbb{Q}^{mm}}\left(\nu^B_{t+2}\left(C_{\bar{T}};T\right)-\frac{1}{\gamma}\mathcal{H}^{me}_{t+1,\bar{T}}\right)-\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t+1,\bar{T}}\right) \tag{4.108}$$

and we are going to show (4.94).

We have

$$V^{C_{\bar{T}}}(x,t) = \sup_{\alpha_{t+1},\ldots,\alpha_{\bar{T}}} E_{\mathbb{P}}\left(-\exp\left(-\gamma \left(x + \Sigma_{i=t}^{\bar{T}-1} \alpha_{i+1}(S_{i+1}-S_i) - C_{\bar{T}} \right)\right) \middle| \mathcal{F}_t \right) \tag{4.109}$$

$$\begin{aligned} &= \sup\nolimits_{\alpha_{t+1}} E_{\mathbb{P}}\left(e^{-\gamma(x+\alpha_{t+1}(S_{t+1}-S_t))} \right. \\ &\sup\nolimits_{\alpha_{t+2},\ldots,\alpha_{\bar{T}}} E_{\mathbb{P}}\left(\left. -e^{-\gamma\left(\Sigma_{i=t+1}^{\bar{T}-1}\alpha_{i+1}(S_{i+1}-S_i)-C_{\bar{T}}\right)} \right| \mathcal{F}_{t+1} \right) \middle| \mathcal{F}_t \right) \end{aligned} \tag{4.110}$$

$$= \sup_{\alpha_{t+1}} E_{\mathbb{P}}\left(e^{-\gamma(x+\alpha_{t+1}(S_{t+1}-S_t))} U_{t+1}^{B}\left(-\nu_{t+1}^{B}(C_{\bar{T}};T) \right) \middle| \mathcal{F}_t \right) \tag{4.111}$$

$$= \sup_{\alpha_{t+1}} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(x + \alpha_{t+1}(S_{t+1}-S_t) - \nu_{t+1}^{B}(C_{\bar{T}};T) \right) - \mathcal{H}_{t+1,\bar{T}}^{me} \right) \middle| \mathcal{F}_t \right) \tag{4.112}$$

$$\begin{aligned} &= \sup\nolimits_{\alpha_{t+1}} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(x + \alpha_{t+1}(S_{t+1}-S_t) \right.\right.\right. \\ &\left.\left.\left. -\left(\nu_{t+1}^{B}(C_{\bar{T}};T) - \tfrac{1}{\gamma}\mathcal{H}_{t+1,\bar{T}}^{me} \right)\right)\right) \middle| \mathcal{F}_t \right) \end{aligned} \tag{4.113}$$

$$= -\exp\left(-\gamma\left(x - \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(\nu_{t+1}^{B}(C_{\bar{T}};T) - \frac{1}{\gamma}\mathcal{H}_{t+1,\bar{T}}^{me} \right)\right) - h_{t+1} \right), \tag{4.114}$$

where we used the appropriate single-period arguments. Using once again proposition 11 of chapter 3 and (3.28), we have

$$V^{0}(x,t) = -\exp\left(-\gamma x - \left(-\gamma \mathcal{E}_{\mathbb{Q}^{mm}}^{(t,t+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{t+1,\bar{T}}^{me} \right) + h_{t+1} \right)\right) \tag{4.115}$$

and using (4.90) we conclude. ■

Next we produce the pricing algorithm in terms of the minimal entropy measure.

Theorem 22 *Let $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$ be the claim to be priced. Let $\mathbb{Q}^{me}$ be the minimal entropy measure and $\mathcal{H}_{t,\bar{T}}^{me}$ be the aggregate entropy. The following statements are true:*

(i) The indifference price $\nu_t^B(C_{\bar{T}};T)$, defined in (4.90), is given by the algorithm

$$\nu_{\bar{T}}^{B}(C_{\bar{T}};T) = C_{\bar{T}}, \tag{4.116}$$

$$\nu_t^{B}(C_{\bar{T}};T) = \mathcal{E}_{\mathbb{Q}^{me}}^{(t,t+1)}\left(\nu_{t+1}^{B}(C_{\bar{T}};T) \right), \tag{4.117}$$

where $\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{me}}$ is the single-step price functional defined in (2.9).

(ii) The indifference price process is given by

$$\nu^B_t(C_{\bar{T}};T) = \mathcal{E}^{(t,\bar{T})}_{\mathbb{Q}^{me}}\left(\nu_{t+1}\left(C_{\bar{T}};T\right)\right), \quad t = 0, 1, ...\bar{T}, \tag{4.118}$$

with the multi-step price functional $\mathcal{E}^{(t,\bar{T})}_{\mathbb{Q}^{me}}$ defined in (2.8).

(iii) The pricing algorithm is consistent across time in that, for $0 \leq t \leq s \leq \bar{T}$, the semigroup property

$$\nu^B_t\left(C_{\bar{T}};T\right) = \mathcal{E}^{(t,s)}_{\mathbb{Q}^{me}}(\mathcal{E}^{(s,\bar{T})}_{\mathbb{Q}^{me}}\left(C_{\bar{T}}\right)) = \mathcal{E}^{(t,s)}_{\mathbb{Q}^{me}}((\nu^B_s\left(C_{\bar{T}};T\right)) = \nu^B_t(\mathcal{E}^{(s,\bar{T})}_{\mathbb{Q}^{me}}(C_{\bar{T}};T)) \tag{4.119}$$

holds.

(iv) The value function $V^{C_{\bar{T}}}$, defined in (2.4), can be written in the form

$$V^{C_{\bar{T}}}\left(x,t\right) = -\exp\left(-\gamma\left(x - \nu^B_t\left(C_{\bar{T}};T\right)\right) - \mathcal{J}^{(t,\bar{T})}_{\mathbb{Q}^{me}}\left(\sum_{i=t+1}^{\bar{T}} h_i\right)\right), \tag{4.120}$$

with $\mathcal{J}^{(t,\bar{T})}_{\mathbb{Q}^{me}}$ as in (3.27).

Proof. Using $Z = \gamma\nu^B_{t+1}\left(C_{\bar{T}};T\right) - \mathcal{H}^{me}_{t+1,\bar{T}}$ in (3.28) yields

$$\begin{aligned}&\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(\gamma\nu_{t+1}\left(C_{\bar{T}}\right) - \mathcal{H}^{me}_{t+1,\bar{T}}\right) = \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}}\left(\gamma\nu^B_{t+1}\left(C_{\bar{T}};T\right)\right) + \\ &\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(-\mathcal{H}^{me}_{t+1,\bar{T}}\right)\end{aligned} \tag{4.121}$$

and, in turn,

$$\begin{aligned}&\tfrac{1}{\gamma}\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(\gamma\nu^B_{t+1}\left(C_{\bar{T}};T\right) - \mathcal{H}^{me}_{t+1,\bar{T}}\right) = \tfrac{1}{\gamma}\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}}\left(\gamma\nu^B_{t+1}\left(C_{\bar{T}};T\right)\right) + \\ &\tfrac{1}{\gamma}\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(-\mathcal{H}^{me}_{t+1,\bar{T}}\right).\end{aligned} \tag{4.122}$$

Thus

$$\mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(\nu^B_{t+1}\left(C_{\bar{T}};T\right) - \frac{\mathcal{H}^{me}_{t+1,\bar{T}}}{\gamma}\right) = \mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{me}}\left(\nu^B_{t+1}\left(C_{\bar{T}};T\right)\right) + \mathcal{E}^{(t,t+1)}_{\mathbb{Q}^{mm}}\left(-\frac{\mathcal{H}^{me}_{t+1,\bar{T}}}{\gamma}\right) \tag{4.123}$$

and we easily conclude. ■

Next we focus on properties of the indifference prices in terms of their payoffs and risk aversion parameter. When stating results regarding the risk aversion, we occasionally use the modified notation $\mathcal{E}_Q^{t,u}(\cdot;\gamma)$ instead of $\mathcal{E}_Q^{t,u}(\cdot;T)$ to reference a specific level of γ. The backward normalization point T is fixed and is the same for all results considered in this section.

Proposition 4.2 *The function $\gamma \to \nu_t^B(C_{\bar{T}};T)$ from R_+ into R is decreasing and continuous.*

Proof. We will use the price representation (4.118). Continuity follows directly from the formula (4.117) and the continuity of the single-step functionals $\mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}(\cdot)$. To establish monotonicity, let us assume that $0<\gamma_1<\gamma_2$. We first show that

$$\nu_{\bar{T}-1}^B(C_{\bar{T}};\gamma_1) > \nu_{\bar{T}-1}^B(C_{\bar{T}};\gamma_2). \tag{4.124}$$

Holder's inequality yields

$$E_{\mathbb{Q}^{me}}\left(e^{-\gamma_1 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right) \le \left(E_{\mathbb{Q}}\left(e^{-\gamma_2 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right)\right)^{\gamma_1/\gamma_2} \tag{4.125}$$

and, in turn,

$$-\frac{1}{\gamma_1}\ln E_{\mathbb{Q}^{me}}\left(e^{-\gamma_1 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right) \ge -\frac{1}{\gamma_2}\ln E_{\mathbb{Q}^{me}}\left(e^{-\gamma_2 C_{\bar{T}}}\left|\mathcal{F}_{\bar{T}-1}\vee\mathcal{F}_{\bar{T}}^S\right.\right). \tag{4.126}$$

Taking the expectation with respect to $\mathbb{Q}^{me}$, we deduce (4.124). The rest of the proof follows by straightforward induction arguments. ■

Proposition 4.3 *The following limiting relations hold*

$$\lim_{\gamma\to 0^+}\nu_t^B(C_{\bar{T}};\gamma) = E_{\mathbb{Q}^{me}}\left(C_{\bar{T}}\left|\mathcal{F}_t\right.\right), \tag{4.127}$$

and

$$\lim_{\gamma\to\infty}\nu_t^B(C_{\bar{T}};\gamma) = \nu_t^{B,Inf}(C), \tag{4.128}$$

with

$$\begin{cases} \nu_{\bar{T}}^{B,Inf}(C) = C_{\bar{T}}, \\ \nu_t^{B,Inf}(C) = E_{\mathbb{Q}^{me}}\left(\inf_{Y_{t+1}}\nu_{t+1}^{B,Inf}(C)\right), t<\bar{T}. \end{cases} \tag{4.129}$$

Proof. We recall that

$$\nu_t^B(C_{\bar{T}};\gamma)=\mathcal{E}_{\mathbb{Q}^{me}}^{(t,t+1)}\left(\mathcal{E}_{\mathbb{Q}^{me}}^{(t+1,t+2)}\left(\left(...\mathcal{E}_{\mathbb{Q}^{me}}^{(\bar{T}-1,\bar{T})}\left(C_{\bar{T}}\right)\right)\right)\right), \tag{4.130}$$

with

$$\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(\cdot)=-E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}^{me}}\left(e^{-\gamma(\cdot)}\left|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right.\right)|\mathcal{F}_s\right) \tag{4.131}$$

for $s=t,t+1,..,\bar{T}$. For convenience, we denote by t_i the intermediate points of $[t,\bar{T}]$, $t=t_0\leq...\leq t_{n-1}\leq\bar{T}=t_n$. We define

$$\mathcal{E}_{\mathbb{Q}^{me}}^{(t_i,t_{i+1})}(\cdot\ ;\ \gamma_{i+1}) \tag{4.132}$$

$$=-E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma_{i+1}}\log E_{\mathbb{Q}^{me}}\left(e^{-\gamma_{i+1}(\cdot)}\left|\mathcal{F}_{t_i}\vee\mathcal{F}_{t_{i+1}}^S\right.\right)|\mathcal{F}_{t_{i+1}}\right) \tag{4.133}$$

for $i=0,1,...,n-1$. Thus, the indifference price can be written as

$$\nu_t^B(C_{\bar{T}};\gamma_1,...\gamma_n) \tag{4.134}$$

$$=\mathcal{E}_{\mathbb{Q}^{me}}^{(t_0,t_1)}\left(\mathcal{E}_{\mathbb{Q}^{me}}^{(t_1,t_2)}\left(\left(...\mathcal{E}_{\mathbb{Q}^{me}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right)\right);\gamma_2\right);\gamma_1\right) \tag{4.135}$$

with $\gamma_i=\gamma$ for $i=1,..,n$. Using a classical diagonalization argument, we deduce that

$$\lim_{\gamma\to\bar{\gamma}}\nu_t^B(C_{\bar{T}};\gamma)=\lim_{\gamma_1\to\bar{\gamma}}\left(...\left(\lim_{\gamma_n\to\bar{\gamma}}\nu_t^B(C_{\bar{T}};\gamma_1,...\gamma_n)\right)\right) \tag{4.136}$$

$$=\lim_{\gamma_1\to\bar{\gamma}}\mathcal{E}_{\mathbb{Q}^{me}}^{(t_0,t_1)}\left(\lim_{\gamma_2\to\bar{\gamma}}\mathcal{E}_{\mathbb{Q}^{me}}^{(t_1,t_2)}\left(\left(...\lim_{\gamma_n\to\bar{\gamma}}\mathcal{E}_{\mathbb{Q}^{me}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right)\right);\gamma_2\right);\gamma_1\right) \tag{4.137}$$

The limits in (4.127) and (4.128) then correspond to the cases $\bar{\gamma}=0$ and $\bar{\gamma}=\infty$.

We start with the case $\bar{\gamma}=0$. Clearly, the analysis reduces to the specification of the individual limits as $\gamma_{m+1}\to 0$ of the nested single-step price functionals $\mathcal{E}_{\mathbb{Q}^{me}}^{(t_m,t_{m+1})}\left(\cdot;\gamma_{m+1}\right)$, for $m=0,1,..,n-1$. We first look at

$$\lim_{\gamma_n\to 0}\mathcal{E}_{\mathbb{Q}^{me}}^{(t_{n-1},t_n)}\left(C_{\bar{T}};\gamma_n\right) \tag{4.138}$$

$$= \lim_{\gamma_n \to 0} -E_{\mathbb{Q}^{me}} \left(\frac{1}{\gamma_n} \log E_{\mathbb{Q}^{me}} \left(e^{-\gamma_n C_{\bar{T}}} \left| \mathcal{F}_{\bar{T}-1} \vee \mathcal{F}^S_{\bar{T}} \right) | \mathcal{F}_{\bar{T}} \right) \tag{4.139}$$

where we remind the reader that $\bar{T} - 1 = t_{n-1}$ and $\bar{T} = t_n$. Recall that the last term coincides with the indifference price $\nu_{\bar{T}-1}(C_{\bar{T}}; \gamma_n)$ and it can be computed explicitly. The expression above expands as

$$\nu^B_{\bar{T}-1}(C_{\bar{T}}; \gamma_n) = \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{me}}(C_{\bar{T}}) = -$$

$$\frac{\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\gamma_n} \log \left(\frac{\mathbb{Q}^{me}(A_{\bar{T}} B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} e^{-\gamma_n C^{uu}_{\bar{T}}} + \frac{\mathbb{Q}^{me}(A_{\bar{T}} B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} e^{-\gamma_n C^{ud}_{\bar{T}}} \right) -$$

$$\frac{\mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\gamma_n} \log \left(\frac{\mathbb{Q}^{me}(A^c_{\bar{T}} B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} e^{-\gamma_n C^{du}_{\bar{T}}} + \frac{\mathbb{Q}^{me}(A^c_{\bar{T}} B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} e^{-\gamma_n C^{dd}_{\bar{T}}} \right), \tag{4.140}$$

where events $A_{\bar{T}}$, $A^c_{\bar{T}}$, $B_{\bar{T}}$, and $B^c_{\bar{T}}$ were defined in equation (3.4) of Chapter 3 and represent the four possible outcomes for the time $\bar{T}$ state space, as seen from time $\bar{T} - 1$.

Taking $\gamma_n \to 0$ yields

$$\begin{aligned} &\lim\nolimits_{\gamma_n \downarrow 0} \mathcal{E}^{(t_{n-1},t_n)}_{\mathbb{Q}^{me}} (C_{\bar{T}}; \gamma_n) = \mathbb{Q}^{me}(A_{\bar{T}} B_{\bar{T}}/\mathcal{F}_{\bar{T}-1}) C^{uu}_{\bar{T}} \\ &+ \mathbb{Q}^{me}(A_{\bar{T}} B^C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}) C^{ud}_{\bar{T}} + \mathbb{Q}^{me}(A^c_{\bar{T}} B_{\bar{T}}/\mathcal{F}_{\bar{T}-1}) C^{du}_{\bar{T}} + \mathbb{Q}^{me}(A^c_{\bar{T}} B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1}) C^{dd}_{\bar{T}}. \end{aligned} \tag{4.141}$$

Combining the above yields

$$\lim_{\gamma_n \to 0} \mathcal{E}^{(t_{n-1},t_n)}_{\mathbb{Q}^{me}} (C_{\bar{T}}; \gamma_n) = E_{\mathbb{Q}^{me}}(C_{\bar{T}} \left| \mathcal{F}_{\bar{T}-1} \right.). \tag{4.142}$$

Repeating the arguments as $\gamma_{n-1} \to 0, ..., \gamma_1 \to 0$, we deduce that

$$\lim_{\gamma \to 0} \nu^B_t(C_{\bar{T}}; \gamma) = E_{\mathbb{Q}^{me}} \left(\left(...E_{\mathbb{Q}^{me}} \left(E_{\mathbb{Q}^{me}}(C_{\bar{T}} \left| \mathcal{F}_{\bar{T}-1} \right.) \left| \mathcal{F}_{\bar{T}-2} \right. \right) \right) | \mathcal{F}_t \right) \tag{4.143}$$

and using the law of iterative expectations we obtain (4.127).

Next, we consider the case $\bar{\gamma} = \infty$ and proceed in a manner similar to the above arguments. We start with the last limit, namely, $\lim_{\gamma \to \infty} \mathcal{E}^{(t_{n-1},t_n)}_{\mathbb{Q}^{me}} (C_{\bar{T}}; \gamma_n)$ for which we observe

$$\lim_{\gamma_n \to \infty} \mathcal{E}^{(t_{n-1},t_n)}_{\mathbb{Q}^{me}} (C_{\bar{T}}; \gamma_n) \tag{4.144}$$

$$= \lim_{\gamma_n \to \infty} -E_{\mathbb{Q}^{me}} \left(\frac{1}{\gamma_n} \log E_{\mathbb{Q}^{me}} \left(e^{-\gamma_n C_{\bar{T}}} \left| \mathcal{F}_{\bar{T}-1} \vee \mathcal{F}_{\bar{T}}^S \right) \right| \mathcal{F}_{\bar{T}-1} \right) \tag{4.145}$$

$$= q_{\bar{T}} \min \left(C \left(S_{\bar{T}}^u, Y_{\bar{T}}^u \right), C \left(S_{\bar{T}}^u, Y_{\bar{T}}^d \right) \right) + (1 - q_{\bar{T}}) \min \left(C \left(S_{\bar{T}}^u, Y_{\bar{T}}^u \right), C \left(S_{\bar{T}}^d, Y_{\bar{T}}^d \right) \right) \tag{4.146}$$

Using the continuity of the involved single-step pricing functionals,

$$\lim_{\gamma_{n-1} \to \infty} \mathcal{E}_{\mathbb{Q}^{me}}^{(t_{n-2}, t_{n-1})} \left(\lim_{\gamma_n \to \infty} \mathcal{E}_{\mathbb{Q}}^{(t_{n-1}, t_n)} \left(C_{\bar{T}}; \gamma_n \right); \gamma_{n-1} \right) \tag{4.147}$$

$$= \lim_{\gamma_{n-1} \to \infty} \mathcal{E}_{\mathbb{Q}^{me}}^{(t_{n-2}, t_{n-1})} \left(E_{\mathbb{Q}^{me}} (\inf_{Y_{\bar{T}}} C_{\bar{T}} / \mathcal{F}_{\bar{T}-1}); \gamma_{n-1} \right) \tag{4.148}$$

As $\gamma_{n-1} \to \infty$, $\nu_t^B(C_{\bar{T}}; \gamma) \downarrow \nu_t^{B,Inf}(C)$. Working similarly with the rest of single-period limits, we obtain 4.128 ■

Next, we explore monotonicity, convexity, and scaling behavior of the indifference prices. We note that all inequalities below hold almost surely under the historical and minimal entropy measures. Since these measures are equivalent, we skip any measure-specific notation.

Proposition 4.4 *The following statements hold:*

(i) The indifference price is a non-decreasing function of the claim's payoff, namely,

$$\text{if } C_{\bar{T}}^1 \leq C_{\bar{T}}^2 \quad \text{then } \; \nu_t^B \left(C_{\bar{T}}^1; T \right) \leq \nu_t^B \left(C_{\bar{T}}^2; T \right). \tag{4.149}$$

In addition, for $\alpha \in (0, 1)$ and $C \geq 0$,

$$\nu_t^B \left(\alpha C_{\bar{T}}^1 + (1 - \alpha) C_{\bar{T}}^2; T \right) \geq \alpha \nu_t^B \left(C_{\bar{T}}^1; T \right) + (1 - \alpha) \nu_t^B \left(C_{\bar{T}}^2; T \right), \tag{4.150}$$

(ii) Let t_i, $i = 0, 1, .., n$ with $t = t_0 \leq t_1 \leq ... \leq t_n = \bar{T}$. Then, for $C \geq 0$, the indifference price satisfies

$$\nu_t^B \left(\alpha C_{\bar{T}}; T \right) \geq \alpha^n \nu_t^B \left(C_{\bar{T}}; T \right) \quad \text{for } \alpha \in (0, 1) \tag{4.151}$$

and

$$\nu_t^B \left(\alpha C_{\bar{T}}; T \right) \leq \alpha^n \nu_t^B \left(C_{\bar{T}}; T \right) \quad \text{for } \alpha \geq 1. \tag{4.152}$$

Proof. Monotonicity (4.149) follows directly from elementary backward induction arguments and monotonicity of the single-period functionals $\mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}(\cdot)$ To establish

(4.150), we show it first for $t = \bar{T} - 1$. Applying Holder's inequality to obtain

$$\begin{aligned}
&-E_{\mathbb{Q}^{me}}\left(\tfrac{1}{\gamma}\ln E_{\mathbb{Q}^{me}}\left(e^{-\gamma\left(\alpha C^1_{\bar{T}}+(1-\alpha)C^2_{\bar{T}}\right)}\left|S_{\bar{T}}\right.\right)\right) = \\
&\geq -E_{\mathbb{Q}^{me}}\left(\tfrac{1}{\gamma}\ln\left(\left(E_{\mathbb{Q}^{me}}\left(e^{-\gamma C^1_{\bar{T}}}\left|S_{\bar{T}}\right.\right)\right)^{\alpha}\left(E_{\mathbb{Q}^{me}}\left(e^{-\gamma C^2_{\bar{T}}}\left|S_{\bar{T}}\right.\right)\right)^{1-\alpha}\right)\right) \\
&= -\alpha E_{\mathbb{Q}^{me}}\left(\tfrac{1}{\gamma}\ln E_{\mathbb{Q}^{me}}\left(e^{-\gamma C^1_{\bar{T}}}\left|S_{\bar{T}}\right.\right)\right) \\
&-(1-\alpha)\,E_{\mathbb{Q}^{me}}\left(\tfrac{1}{\gamma}\ln E_{\mathbb{Q}^{me}}\left(e^{-\gamma C^2_{\bar{T}}}\left|S_{\bar{T}}\right.\right)\right).
\end{aligned} \tag{4.153}$$

To show (4.150) for $t < \bar{T} - 1$, we use induction and (4.117). Inequalities (4.151) and (4.152) follow using induction, the appropriate single-period arguments and (4.117). ■

So far we have presented the indifference valuation algorithms for both the backward and forward prices. The forward indifference valuation involves the minimal martingale measure $\mathbb{Q}^{mm}$ and the nonlinear pricing functionals $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(\cdot)$. The backward indifference valuation shares some of the forward valuation features. Both of the algorithms are recursive, proceed backwards in time, and structurally are written in terms of the same nonlinear pricing functionals $\mathcal{E}^{t,t+1}_{Q}(\cdot)$. However, there are some differences. Theorem 22 states that the backward valuation uses the minimal entropy measure $\mathbb{Q}^{me}$, and not the $\mathbb{Q}^{mm}$. The valuation with backward utility could be done using the minimal martingale measure $\mathbb{Q}^{mm}$ (see Theorem 21), but would then use the nonlinear functionals $\mathcal{P}^{t,t+1}_{Q}(\cdot)$ (defined in equation (4.91)) whose structure is different from $\mathcal{E}^{t,t+1}_{Q}(\cdot)$ (defined in Chapter 2 equation (2.9)). The nonlinear functionals $\mathcal{E}^{t,t+1}_{Q}(\cdot)$ do not carry any dependence on either the forward normalization time point s, or the backward normalization point T, and neither does the minimal martingale measure $\mathbb{Q}^{mm}$. The minimal entropy measure $\mathbb{Q}^{me}$ does depend on the backward normalization point T, which also represents the end of the investment horizon. The minimal entropy measure transfers its T-dependence onto the backward indifference prices it generates. Dependence on the end of the investment horizon T is another important difference between the backward and forward prices.

4.3 The forward and the backward dynamic valuation in the reduced model

In this section we address two important issues. First, we provide conditions under which the two dynamic utilities yield identical indifference prices. Clearly, they do so when the market is complete, but this is not the only case. Second, we address the reduced model properties of the forward and backward prices with respect to hedgeable risks.

The concept of a reduced model was introduced in chapter 3. In such a model the historical distribution of the next period's traded asset value does not depend of the path of the non-traded risk factor Y up to the current time t, and neither do the variables ξ_{t+1}^{u} and ξ_{t+1}^{d}, for all $0 \leq t \leq T-1$. We have shown in corollary 10 of chapter 3 that in a reduced model, the minimal martingale measure and the minimal entropy measure are the same. A natural consequence of that corollary is that, in the reduced model, the backward and forward indifference prices are the same as well.

Theorem 23 *In the reduced binomial model, i.e. when*

$$\mathbb{P}(\xi_{t+1}/\mathcal{F}_t) = \mathbb{P}(\xi_{t+1}/\mathcal{F}_t^S),\ t = 0, 1, ..., T-1. \tag{4.154}$$

and ξ_{t+1}^{u}, ξ_{t+1}^{d} *are* $\mathcal{F}_t^S$*-measurable, the forward and backward indifference prices of the European contract* C *written at* t_0 *and maturing at* $\bar{T}$*, coincide:*

$$\nu_t^F\left(C_{\bar{T}}\right) = \nu_t^B\left(C_{\bar{T}}; T\right) \tag{4.155}$$

for $t_0 \leq t \leq \bar{T} \leq T$.

The proof of the above result follows easily from corollary 10 of Chapter 3, the algorithms are derived for the forward and the backward prices (Theorems 14 and 21, correspondingly).

The next result assumes the reduced model and considers the payoff depending on the traded asset only. In this case both the backward and forward prices coincide with the complete market price. The proof uses $\mathcal{F}_{\bar{T}}^S$-measurability of the payoff, the independence of ξ_{t+1}^{u} and ξ_{t+1}^{d} of the stochastic factor Y, and follows easily from the valuation algorithms of Theorems 21 and 14.

Proposition 4.5 *Under the reduced model assumption of corollary 10, Chapter 3 (also shown in Theorem 23 above), if the payoff* $C(S_{\bar{T}}, Y_{\bar{T}})$ *does not depend on* $Y_{\bar{T}}$*, then*

$$\nu_t^F(C_{\bar{T}}) = \nu_t^B(C_{\bar{T}}; T) = E_Q(C(S_{\bar{T}})/\mathcal{F}_t^S), \tag{4.156}$$

for any martingale measure Q *equivalent to* $\mathbb{P}$ *and any* $t_0 \leq t \leq \bar{T}$*.*

One example of a reduced form model would be when ξ_t^{u+1} and ξ_{t+1}^d are constants and the property (4.154) holds. The above assumptions are the ones used by [41], and are very common in the literature.

Chapter 5

Dynamic indifference valuation of American contracts in the discrete model.

5.1 Dynamic indifference valuation with the forward utility.

In this section we show how liabilities with early exercise can be valued in the discrete time framework using the forward dynamic utility. We formulate the corresponding value functions for the agent as stochastic control problems incorporating both the agent's investment policy and the exercise time of the claim. We define and derive closed-form formulas for the indifference price. We investigate properties of the indifference price with respect to risk aversion, in particular its monotonicity and limiting values as risk aversion becomes infinitely small or infinitely large. We provide a characterization of the optimal exercise time and the optimal hedging policy, and derive a new representation for the early exercise indifference price. The new representation relates the price of the American contract to prices of European claims of maturities within the allowed exercise horizon. We extend the algorithm to partial exercise in subsequent chapters. We remind the reader that the name "forward" is not being used in the traditional way of referring to wealth being expressed in forward units. Herein, the name "forward" refers to the way the dynamic utility propagates, from initial point s, at which it is fixed, forward in time.

Let t be the current valuation time, $\alpha = (\alpha_{t+1}, ..., \alpha_{\bar{T}})$ be a trading strategy and τ be a stopping time (to exercise the claim) chosen by the investor. Here we consider the early exercise only on a finite horizon and thus we require $t \leq \tau \leq \bar{T}$ a.s. We also require that the claim be exercised at $\bar{T}$ if it has not been exercised before that time. The early exercise liability yields intrinsic payoff C_τ, if exercised at time τ. The total wealth of the investor time τ combines proceeds X_τ accumulated through trading with the payoff upon exercise C_τ. At time τ the agent measures his total wealth through the forward dynamic utility $U^F(X_\tau + C_\tau, \tau; s)$, normalized at s, $s \leq t$. Investor's time t value function is defined accordingly:

Definition 5.1 *The time t value function of the buyer of the American claim C, written at t_0 and maturing at $\bar{\bar{T}}$, is defined as the expected utility of his total wealth upon exercise, optimized through the choice of the trading strategy α and of the stopping time τ. Namely,*

$$V^C(X_t, S_t, Y_t, t; s) = \sup_{\alpha_{t+1},\dots,\alpha_{\bar{T}}} \sup_{t \leq \tau \leq \bar{T}} E_{\mathbb{P}}[U_\tau^F(X_\tau + C_\tau; s)/\mathcal{F}_t]. \tag{5.1}$$

Given the notion of the buyer's value function, we define the indifference price as:

Definition 5.2 *Let s be the forward normalization point. The buyer's indifference price of the American claim C, written at t_0 and maturing at $\bar{\bar{T}}$, is the amount $a_t(C; s)$ to be payed for the claim, which makes the value functions V^C equal to U^F. Namely, $a_t(C; s)$ is the amount satisfying:*

$$V^C(X_t, S_t, Y_t, t; s) = U_t^F(X_t + a_t(C; s); s). \tag{5.2}$$

The two definitions above, in analogy with the way we defined the indifference price for European contracts, can easily be combined into just one pricing condition:

$$U_t^F(X_t + a_t(C; s); s) = \sup_{\alpha_{t+1},\dots,\alpha_{\bar{T}}} \sup_{t \leq \tau \leq \bar{T}} E_{\mathbb{P}}[U_\tau^F(X_\tau + C_\tau; s)/\mathcal{F}_t]. \tag{5.3}$$

This way, the concept of value function will no longer be needed anymore, as is the case with European claims. For convenience, we choose to keep the definition of the value function for convenience, so that the relevant results regarding the righthand side of (5.3) can be formulated more simply.

We are aiming at constructing the recursive algorithm that would relate the time t indifference price to the one of time $t+1$. Indifference prices are obtained by studying the corresponding value functions. Thus, one should proceed by relating the buyer's value function of time t to that one of time $t+1$. In the discrete time model, the claim can either be exercised immediately, or the investor will have to wait till the next period to be able to exercise the claim. Using this observation we arrive at a simplified expression for the value function.

Theorem 24 *If the optimal $\tau_u^\star$ and $\alpha_{u+1}^\star, ..., \alpha_{\bar{T}}^\star$ solving 5.1 exists for all $t \leq u \leq \bar{T}$, then $V^C(X_t, S_t, Y_t, t; s)$ has the following recursive representation:*

$$\begin{cases} V^C(X_t, S_t, Y_t, t; s) = \\ \max \left\{ U_t^F(X_t + C_t; s), \sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; s)/\mathcal{F}_t] \right\}, \\ V^C(X_{\bar{T}}, S_{\bar{T}}, Y_{\bar{T}}, \bar{T}; s) = U_{\bar{T}}^F(X_{\bar{T}} + C_{\bar{T}}; s). \end{cases} \tag{5.4}$$

Proof. If optimizers $\tau_u^\star$ and $\alpha^\star$ exist for all $t \leq u \leq \bar{T}$,

$$V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; s) = E_{\mathbb{P}}[U_{\tau_{t+1}^\star}^F(X_{\tau_{t+1}^\star} + C_{\tau_{t+1}^\star}; s)/\mathcal{F}_{t+1}]. \tag{5.5}$$

$\tau_{t+1}^\star$ and $\alpha_{t+2}^\star, ..., \alpha_{\bar{T}}^\star$ may not be optimal solutions of (5.1) at time t, implying for any choice of α_{t+1} that $V^C(X_t, S_t, Y_t, t; s) \geq E_{\mathbb{P}}[U_{\tau_{t+1}^\star}^F(X_{\tau_{t+1}^\star} + C_{\tau_{t+1}^\star}; s)/\mathcal{F}_t]$, for $X_{\tau_{t+1}^\star} = X_t + \alpha_{t+1}(S_{t+1} - S_t) + \sum_{u=t+2}^{\tau_{t+1}^\star} \alpha_u^\star(S_u - S_{u-1})$. Conditioning on $\mathcal{F}_{t+1}$, we get

$$\begin{aligned} &V^C(X_t, S_t, Y_t, t; s) \geq E_{\mathbb{P}}[E_{\mathbb{P}}[U_{\tau_{t+1}^\star}^F(X_{\tau_{t+1}^\star} + C_{\tau_{t+1}^\star}; s)/\mathcal{F}_{t+1}]/\mathcal{F}_t] = \\ &E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; s)/\mathcal{F}_t], \end{aligned} \tag{5.6}$$

for $X_{t+1} = X_t + \alpha_{t+1}(S_{t+1} - S_t)$. Since α_{t+1} is arbitrary,

$$V^C(X_t, S_t, Y_t, t; s) \geq \sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; s)/\mathcal{F}_t]. \tag{5.7}$$

Clearly, $V^C(X_t, S_t, Y_t, t; s) \geq U_t^F(X_t + C_t; s)$ and thus $V^C(X_t, S_t, Y_t, t; s)$ is at least as big as the righthand side of (5.4). On the other hand, for any τ and α,

$$\begin{aligned}
&E_{\mathbb{P}}[U_{\tau}^{F}(X_{\tau}+C_{\tau};s)/\mathcal{F}_t] = \\
&\mathbf{1}_{\{\tau=t\}}U_t^F(X_t+C_t;s)+(1-\mathbf{1}_{\{\tau=t\}})E_{\mathbb{P}}[U_{\tau}^{F}(X_{\tau}+C_{\tau};s)/\mathcal{F}_t] = \\
&\mathbf{1}_{\{\tau=t\}}U_t^F(X_t+C_t;s)+(1-\mathbf{1}_{\{\tau=t\}})E_{\mathbb{P}}[E_{\mathbb{P}}[U_{\tau}^{F}(X_{\tau}+C_{\tau};s)/\mathcal{F}_{t+1}]/\mathcal{F}_t] \leq \\
&\mathbf{1}_{\{\tau=t\}}U_t^F(X_t+C_t;s)+(1-\mathbf{1}_{\{\tau=t\}})E_{\mathbb{P}}[V^C(X_{t+1},S_{t+1},Y_{t+1},t+1;s)/\mathcal{F}_t] \leq \\
&\mathbf{1}_{\{\tau=t\}}U_t^F(X_t+C_t;s)+(1-\mathbf{1}_{\{\tau=t\}})\sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1},S_{t+1},Y_{t+1},t+1;s)/\mathcal{F}_t] \leq \\
&\mathbf{1}_{\{\tau=t\}}\max\left\{U_t^F(X_t+C_t;s),\sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1},S_{t+1},Y_{t+1},t+1;s)/\mathcal{F}_t]\right\}+ \\
&(1-\mathbf{1}_{\{\tau=t\}})\max\left\{U_t^F(X_t+C_t;s),\sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1},S_{t+1},Y_{t+1},t+1;s)/\mathcal{F}_t]\right\} = \\
&\max\left\{U_t^F(X_t+C_t;s),\sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1},S_{t+1},Y_{t+1},t+1;s)/\mathcal{F}_t]\right\}
\end{aligned} \tag{5.8}$$

■

To characterize the buyer's indifference price of the American claim C, we introduce another family of operators $\mathcal{A}_Q^{t,u}(Z)$ for $t \leq u \leq \bar{T}$. We let Z be an $\mathcal{F}_u$ - measurable random variable and Q be a martingale measure equivalent to $\mathbb{P}$. Define $\mathcal{A}_Q^{t,u}(Z)$ as

$$\begin{cases} \mathcal{A}_Q^{t,u}(Z) = \mathcal{A}_Q^{t,u-1}\left(\mathcal{A}_Q^{u-1,u}(Z)\right), \\ \mathcal{A}_Q^{t,t}(Z) = Z, \end{cases} \tag{5.9}$$

where

$$\mathcal{A}_Q^{u-1,u}(Z) = \max\left\{C_{u-1},\mathcal{E}_Q^{u-1,u}(Z)\right\}. \tag{5.10}$$

with $\mathcal{E}_Q^{u-1,u}(Z)$ defined earlier in equation (2.9) of chapter 2.

The theorem below presents the multi-period pricing algorithm, the main result of this section.

Theorem 25 *Let $\mathbb{Q}^{mm}$ be a martingale measure equivalent to $\mathbb{P}$ satisfying, for $t = 0,1,...,\bar{T}$,*

$$\mathbb{Q}^{mm}\left(\eta_{t+1}/\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right) = \mathbb{P}\left(\eta_{t+1}/\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right). \tag{5.11}$$

(i) The indifference price $a_t(C;s)$ as in definition (5.2) satisfies

$$\begin{cases} a_t(C;s) = \max\left\{C_t,\mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}(a_{t+1}(C;s))\right\}, t<\bar{T} \\ a_{\bar{T}}(C;s) = C_{\bar{T}}, \end{cases} \tag{5.12}$$

with $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}$ *defined in equation (2.9) of chapter 2 for* $Q = \mathbb{Q}^{mm}$*.*
(ii) The indifference price process is given by

$$a_t\left(C\right) = \mathcal{A}^{t,\bar{T}}_{\mathbb{Q}^{mm}}\left(C_{\bar{T}}\right), \tag{5.13}$$

with $\mathcal{A}^{t,\bar{T}}_{\mathbb{Q}^{mm}}$ *defined in equations (5.9)-(5.10) for* $Q = \mathbb{Q}^{mm}$*.*
(iii) The pricing algorithm is consistent across time, in that, for $0 \le t \le u \le \bar{T}$*, the semi-group property*

$$a_t\left(C;s\right) = \mathcal{A}^{t,u}_{\mathbb{Q}^{mm}}\left(\mathcal{A}^{u,\bar{T}}_{\mathbb{Q}^{mm}}\left(C_{\bar{T}}\right)\right) = \mathcal{A}^{t,u}_{\mathbb{Q}^{mm}}\left(a_u\left(C;s\right)\right) = a_t\left(\mathcal{A}^{u,\bar{T}}_{\mathbb{Q}^{mm}}\left(C_{\bar{T}}\right);s\right) \tag{5.14}$$

holds.

Formula (5.12) has an intuitive interpretation: C_t equals the amount the buyer gains if he exercises the claim immediately; $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}\left(a_{t+1}\left(C;s\right)\right)$ represents the amount the buyer gains if he continues to hold the claim till the next possible exercise time. $a_t\left(C;s\right)$ is then the maximum of the two available alternatives. Formula (5.12) inherits the remarkable nested structure of the discrete time complete market no-arbitrage prices, rolling backwards from the terminal time $\bar{T}$ to the current time t. As in the complete market case, the price is the maximum of two values, one of which is the intrinsic value of the option, the other the value of the alternative "to continue". However, in our pricing scheme market incompleteness shows itself in a number of ways. One is in the choice of the pricing measure, which is different from the complete market case. The measure $\mathbb{Q}^{mm}$ used throughout is the minimal martingale measure, the same measure used in indifference pricing of European claims with forward preferences. Another consequence of market incompleteness is that the value of the alternative "to continue" is now characterized using the (nonlinear) one-period European buyer's indifference functionals, and not the risk-neutral expectations. Also, the prices depend on the level of the absolute risk aversion. Operators $\mathcal{A}^{t,t+1}_{\mathbb{Q}^{mm}}$ are in general nonlinear but, as will be shown, the nonlinearities go away if the two risky stocks are "perfectly correlated" or in the limiting case where risk aversion approaches zero.

Although the forward dynamic utility process depends on the normalization point, this dependency does not appear anywhere in the formula (5.9) for the nonlinear indifference pricing operator $\mathcal{A}^{t,u}_{\mathbb{Q}^{me}}(\cdot)$. The normalization point s does not affect the measure $\mathbb{Q}^{mm}$ in any way, as discussed in section 4.1. Under the forward dynamic utility, the indifference prices do not depend on the normalization point s either. For early exercise contracts, the prices depend on the expiration date of the contract, but unlike with the backward dynamic utility, the expiration date of the contract does not need to be bounded in time by any artificially chosen "end of investment horizon".

Proof. (i) At time $\bar{T}-1$,

$$\begin{aligned}
&V^C(X_{\bar{T}-1}, S_{\bar{T}-1}, Y_{\bar{T}-1}, \bar{T}-1; s) = \\
&\max\left\{ U^F_{\bar{T}-1}(X_{\bar{T}-1}+C_{\bar{T}-1}; s), \sup_{\alpha_{\bar{T}}} E\left[V^C(X_{\bar{T}}, S_{\bar{T}}, Y_{\bar{T}}, \bar{T}; s)/\mathcal{F}_{\bar{T}-1}\right]\right\} = \\
&\max\left\{ U^F_{\bar{T}-1}(X_{\bar{T}-1}+C_{\bar{T}-1}; s), \sup_{\alpha_{\bar{T}}} E\left[U^F_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}}; s)/\mathcal{F}_{\bar{T}-1}\right]\right\}.
\end{aligned} \tag{5.15}$$

$\sup_{\alpha_{\bar{T}}} E\left[U^F_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}}; s)/\mathcal{F}_{\bar{T}-1}\right]$ can be viewed as time $\bar{T}-1$ value function of an investor with forward preferences holding a claim $C_{\bar{T}}$ expiring at $\bar{T}$. Considering the results already obtained for European claims,

$$\sup_{\alpha_{\bar{T}}} E\left[U^F_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}}; s)/\mathcal{F}_{\bar{T}-1}\right] = U^F_{\bar{T}}(X_{\bar{T}} + \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{mm}}(C_{\bar{T}}); s), \tag{5.16}$$

and

$$\begin{aligned}
&V^C(X_{\bar{T}-1}, S_{\bar{T}-1}, Y_{\bar{T}-1}, \bar{T}-1; s) = \\
&\max\left\{ U^F_{\bar{T}-1}(X_{\bar{T}-1}+C_{\bar{T}-1}; s), U^F_{\bar{T}}(X_{\bar{T}} - \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{mm}}(C_{\bar{T}}); s)\right\} = \\
&U^F_{\bar{T}-1}\left(X_{\bar{T}-1} + \max\left\{ C_{\bar{T}-1}, \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{mm}}(C_{\bar{T}})\right\}; s\right),
\end{aligned} \tag{5.17}$$

implying the corresponding relation for the indifference price.

Following a typical induction argument, assume that the formula holds for times $t+1$ through $\bar{T}$. Now one needs to prove that the formula also holds at time t.

From definition 5.2 of the indifference price,

$$\begin{aligned}
&\sup_{\alpha_{t+1}} E\left[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; s)/\mathcal{F}_t\right] = \\
&\sup_{\alpha_{t+1}} E\left[U^F(X_{t+1} + a_{t+1}(C;s), t+1; s)/\mathcal{F}_t\right] = \\
&\sup_{\alpha_{t+1}} E\left[-\exp\left(-\gamma(X_{t+1} + a_{t+1}(C;s)) + \sum_{k=s+1}^{t+1} h_k\right)/\mathcal{F}_t\right] = \\
&\exp\left(\sum_{k=s+1}^{t} h_k\right) \sup_{\alpha_{t+1}} E\left[-\exp(-\gamma(X_{t+1} + a_{t+1}(C;s)) + h_{t+1})/\mathcal{F}_t\right],
\end{aligned} \tag{5.18}$$

since $\sum_{k=s+1}^{t} h_k$ is F_t-measurable. The last expression can be viewed as the time t one-period value function of an investor with forward utility function who holds European claim paying amount $a_{t+1}(C;s)$ at time $t+1$. Based on the obtained results for the European claims in section 4.1 theorem 14,

$$\begin{aligned}
&\exp\left(\sum_{k=s+1}^{t+1} h_k\right) \sup_{\alpha_{t+1}} E\left[-\exp\left(-\gamma(X_{t+1} + a_{t+1}(C;s))\right)/\mathcal{F}_t\right] = \\
&\exp\left(\sum_{k=s+1}^{t+1} h_k\right) \left(-\exp\left(-\gamma\left(X_t + \mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}\left(a_{t+1}(C;s)\right)\right) - h_{t+1}\right)\right) = \\
&-\exp\left(-\gamma\left(X_t + \mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}\left(a_{t+1}(C;s)\right)\right) + \sum_{k=s+1}^{t} h_k\right) = \\
&U_t^F\left(X_t + \mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}\left(a_{t+1}(C;s)\right); s\right).
\end{aligned} \tag{5.19}$$

Therefore, using formula (5.4) for the buyer's value function,

$$\begin{aligned}
&V^C(X_t, S_t, Y_t, t; s) = \max\left\{U_t^F(X_t + C_t; s), U_t^F\left(X_t + \mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}\left(a_{t+1}(C;s)\right); s\right)\right\} = \\
&U_t^F\left(X_t + \max\left\{C_t, \mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}\left(a_{t+1}(C;s)\right)\right\}; s\right).
\end{aligned} \tag{5.20}$$

From the definition of the buyer's indifference price and the calculation above,

$$a_t\left(C;s\right) = \max\left\{C_t, \mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}\left(a_{t+1}\left(C;s\right)\right)\right\}, \tag{5.21}$$

and the proof of Part (i) is complete.
Parts (ii) and (iii) of theorem 25 follow from (i) together with the definition of the operators $\mathcal{A}_{\mathbb{Q}^{mm}}^{t,u}$ introduced above in equations (5.9) and (5.10) ■

We further consider properties of the buyer's indifference price. The next theorem (theorem 26) shows how the price is affected by changes in the absolute risk aversion. Part (i) of the theorem shows that a more risk averse buyer assigns a smaller value to the claim bearing unhedgeable risk. In the next theorem we use a slightly different notation $a_t\left(C\,;\gamma\right)$ and $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}\left(Z_{t+1}\,;\gamma\right)$. The "extra argument" γ specifies a particular level of risk aversion to which the indifference price corresponds. Part (ii) of the theorem characterizes asymptotic behavior as $\gamma \to 0$. Also, it provides an upper bound for $a_t\left(C\,;\gamma\right)$ for all values of γ, obtained as the indifference price for a buyer with infinitesimally small risk aversion. The limiting value shows significant structural similarity to the no-arbitrage price of the American claim in a complete market. However it uses a different measure $\mathbb{Q}^{mm}$, and not the classical risk-neutral measure. The result of Part (ii) is consistent with familiar asymptotic results for European derivatives in discrete and continuous time. Part (iii) describes asymptotic behavior of the price as $\gamma \to \infty$ and provides a lower bound for $a_t(C;\gamma)$ in the form of the buyer's indifference price for an agent with infinitely large risk aversion.

Theorem 26 (Monotonicity and asymptotic bounds) *(i) For $\gamma_1 \geq \gamma_2$*

$$a_t\left(C;\gamma_1\right) \leq a_t\left(C;\gamma_2\right),\ t \leq \bar{T} \ a.s. \tag{5.22}$$

(ii) Define the sequence of $\mathcal{F}_t$-measurable random variables $\nu^0_t(C)$ as

$$\begin{cases} \nu^0_{\bar{T}}(C) = C_{\bar{T}}, \\ \nu^0_t(C) = \max\left\{C_t, E_{\mathbb{Q}^{mm}}\left[\nu^0_{t+1}(C)/\mathcal{F}_t\right]\right\},\ t < \bar{T}\ . \end{cases} \tag{5.23}$$

Then for any $t \leq \bar{T}$,

$$a_t\left(C;\gamma\right) \nearrow \nu^0_t\left(C\right),\ \text{as } \gamma \to 0. \tag{5.24}$$

(iii) Define the sequence of $\mathcal{F}_t$-measurable random variables $\nu^{inf}_t(C)$ as

$$\begin{cases} \nu^{inf}_{\bar{T}}(C) = C_{\bar{T}}, \\ \nu^{inf}_t(C) = \max\left\{C_t, E_{\mathbb{Q}^{mm}}[\min\limits_{Y_{t+1}} \nu^{inf}_{t+1}/\mathcal{F}_t]\right\}, t < \bar{T}. \end{cases} \tag{5.25}$$

Then

$$a_t(C;\gamma) \searrow \nu^{inf}_t(C),\ \text{as } \gamma \to \infty. \tag{5.26}$$

In (5.25), the expectation can be taken under any equivalent to $\mathbb{P}$ martingale measure Q.

Proof. (i) For $t = \bar{T}$, $a_{\bar{T}}(C;\gamma_1) = a_{\bar{T}}(C;\gamma_2) = C_{\bar{T}}$ and inequality (5.22) holds. For $t < \bar{T}$, one only needs to show that for every $\mathcal{F}_{t+1}$-measurable random variable Z_{t+1},

$$\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(Z_{t+1};\gamma_1) \leq \mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(Z_{t+1};\gamma_2)\,. \tag{5.27}$$

Let $\tilde{\gamma} = -\gamma$,

$$\begin{aligned} &\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(Z_{t+1};\gamma) = E_{\mathbb{Q}^{mm}}\left[\ln\left(E_{\mathbb{Q}^{mm}}\left[\left(e^{Z_{t+1}}\right)^{\tilde{\gamma}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]\right)^{\frac{1}{\tilde{\gamma}}}/\mathcal{F}_t\right] = \\ &E_{\mathbb{Q}^{mm}}\left[\ln\left(E_{\mathbb{Q}^{mm}}\left[\left(H_{t+1}\right)^{\tilde{\gamma}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]\right)^{\frac{1}{\tilde{\gamma}}}/\mathcal{F}_t\right] \end{aligned} \tag{5.28}$$

where $H_{t+1} = e^{Z_{t+1}} > 0$. Due to monotonicity of conditional expectation and logarithmic function it only remains to show that for $\tilde{\gamma_1} \leq \tilde{\gamma_2} < 0$

$$E_{\mathbb{Q}^{mm}}\left[\left(H_{t+1}\right)^{\tilde{\gamma_1}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]^{\frac{1}{\tilde{\gamma_1}}} \leq E_{\mathbb{Q}^{mm}}\left[\left(H_{t+1}\right)^{\tilde{\gamma_2}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]^{\frac{1}{\tilde{\gamma_2}}} \tag{5.29}$$

With $H^{\tilde{\gamma_1}}_{t+1} = G_{t+1}$ and $0 < \hat{\gamma} = \frac{\tilde{\gamma_2}}{\tilde{\gamma_1}} \leq 1$ inequality (5.29) can be rewritten (note that the sign of the new inequality is different because $\frac{1}{\tilde{\gamma_1}}$ is negative):

$$E_{\mathbb{Q}^{mm}}\left[G_{t+1}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right] \geq E_{\mathbb{Q}^{mm}}\left[\left(G_{t+1}\right)^{\hat{\gamma}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]^{\frac{1}{\hat{\gamma}}} \tag{5.30}$$

The last inequality follows from concavity of the function $x^{\hat{\gamma}}$ for $x > 0$, $0 < \hat{\gamma} \leq 1$, and the Jensen's inequality.
(ii) We first show that the formula holds for time $\bar{T} - 1$. For any $t \leq \bar{T}$, including $t = \bar{T} - 1$, $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(C_{\bar{T}})$ could be written explicitly as

$$\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(a_{t+1}) = -$$

$$\frac{\mathbb{Q}^{mm}(A_{t+1}/\mathcal{F}_t)}{\gamma} \log\left(\frac{\mathbb{Q}^{mm}(A_{t+1}B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A_{t+1}/\mathcal{F}_t)} e^{-\gamma a^{uu}} + \frac{\mathbb{Q}^{mm}(A_{t+1}B^c_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A_{t+1}/\mathcal{F}_t)} e^{-\gamma a^{ud}_{t+1}}\right) -$$

$$\frac{\mathbb{Q}^{mm}(A^c_{t+1}/\mathcal{F}_t)}{\gamma} \log\left(\frac{\mathbb{Q}^{mm}(A^c_{t+1}B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A^c_{t+1}/\mathcal{F}_t)} e^{-\gamma a^{du}_{t+1}} + \frac{\mathbb{Q}^{mm}(A^c_{t+1}B^c_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A^c_{t+1}/\mathcal{F}_t)} e^{-\gamma a^{dd}_{t+1}}\right), \tag{5.31}$$

with

$$\begin{aligned}
\mathbb{Q}^{mm}(A_{t+1}/\mathcal{F}_t) &= \mathbb{Q}^{mm}\left(S_{t+1} = S^u_{t+1}/\mathcal{F}_t\right),\\
\mathbb{Q}^{mm}(A^c_{t+1}/\mathcal{F}_t) &= \mathbb{Q}^{mm}\left(S_{t+1} = S^d_{t+1}/\mathcal{F}_t\right),\\
\mathbb{Q}^{mm}(A_{t+1}B_{t+1}/\mathcal{F}_t) &= \mathbb{Q}^{mm}\left(S_{t+1} = S^u_{t+1}, Y_{t+1} = Y^u_{t+1}/\mathcal{F}_t\right),\\
\mathbb{Q}^{mm}(A^c_{t+1}B_{t+1}/\mathcal{F}_t) &= \mathbb{Q}^{mm}\left(S_{t+1} = S^d_{t+1}, Y_{t+1} = Y^u_{t+1}/\mathcal{F}_t\right),\\
\mathbb{Q}^{mm}(A_{t+1}B^c_{t+1}/\mathcal{F}_t) &= \mathbb{Q}^{mm}\left(S_{t+1} = S^u_{t+1}, Y_{t+1} = Y^d_{t+1}/\mathcal{F}_t\right),\\
\mathbb{Q}^{mm}(A^c_{t+1}B^c_{t+1}/\mathcal{F}_t) &= \mathbb{Q}^{mm}\left(S_{t+1} = S^d_{t+1}, Y_{t+1} = Y^d_{t+1}/\mathcal{F}_t\right),
\end{aligned} \tag{5.32}$$

with a^{uu}, a^{ud}_{t+1}, a^{du}_{t+1} and a^{dd}_{t+1} denoting the four possible values of $a_{t+1}(C;s)$, as viewed form time t. For $t = \bar{T} - 1$,

$$a^{ij}_{t+1} = C^{ij} = C(S^i_{\bar{T}}, Y^j_{\bar{T}}), \text{ for } i, j = \text{'}u\text{' and '}d\text{'}\ . \tag{5.33}$$

Using L'Hopital's rule and equation (5.31), the limit of $\mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{mm}}(C_{\bar{T}})$ as $\gamma \to 0$ is

$$\begin{aligned}
&\mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\left(\frac{\mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} C^{uu} + \frac{\mathbb{Q}^{mm}(A_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} C^{ud}\right)\\
&+\mathbb{Q}^{mm}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\left(\frac{\mathbb{Q}^{mm}(A^c_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} C^{du} + \frac{\mathbb{Q}^{mm}(A^c_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{mm}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})} C^{dd}\right)\\
&= E_{\mathbb{Q}^{mm}}\left[C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}\right].
\end{aligned} \tag{5.34}$$

The calculation above shows that $\mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{mm}}(C_{\bar{T}})$ converges to $E_{\mathbb{Q}^{mm}}\left[C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}\right]$. Consequently, since $C_{\bar{T}-1}$ is independent of γ,

$$a_{\bar{T}-1}(C;\gamma) \to \max\{C_{\bar{T}-1}, E_{\mathbb{Q}^{mm}}\left[C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}\right]\},$$

as $\gamma \to 0$.

To confirm the formula for $t \leq \bar{T} - 2$, one would proceed by induction and assume that the limit of $a_u(C;\gamma)$ as $\gamma \to 0$ equals $\nu_u^0(C)$ for all $u \geq t+1$. $a_t(C;\gamma)$ is the maximum of C_t and $\mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}(a_{t+1}(C;\gamma))$. The latter has an explicit representation of equation (5.31). Since, by induction assumption, $a_{t+1}(C;\gamma)$ converges to $\nu_{t+1}^0(C)$ as $\gamma \to 0$, applying the same argument as before yields:

$$\begin{aligned}
&\lim_{\gamma \to 0} \mathcal{E}_{\mathbb{Q}^{mm}}^{t,t+1}(a_{t+1}(C;\gamma)) = \\
&\mathbb{Q}^{mm}(A_{t+1}/\mathcal{F}_t)\left(\frac{\mathbb{Q}^{mm}(A_{t+1}B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A_{t+1}/\mathcal{F}_t)}a_{t+1}^{uu} + \frac{\mathbb{Q}^{mm}(A_{t+1}B_{t+1}^c/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A_{t+1}/\mathcal{F}_t)}a_{t+1}^{ud}\right) + \\
&\mathbb{Q}^{mm}(A_{t+1}^c/\mathcal{F}_t)\left(\frac{\mathbb{Q}^{mm}(A_{t+1}^c B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A_{t+1}^c/\mathcal{F}_t)}a_{t+1}^{du} + \frac{\mathbb{Q}^{mm}(A_{t+1}^c B_{t+1}^c/\mathcal{F}_t)}{\mathbb{Q}^{mm}(A_{t+1}^c/\mathcal{F}_t)}a_{t+1}^{dd}\right) = \\
&E_{\mathbb{Q}^{mm}}\left[\nu_{t+1}^0(C)/\mathcal{F}_t\right].
\end{aligned} \tag{5.35}$$

Thus $a_t(C;\gamma)$ converges to $\max\{C_t, E_{\mathbb{Q}^{mm}}[\nu_{t+1}^0(C)/\mathcal{F}_t]\}$.
(iii) As before we show that formula for $t = \bar{T} - 1$. For other values of $t \leq \bar{T} - 2$ one can use induction and repeat the argument. The limit of $\mathcal{E}_{\mathbb{Q}^{mm}}^{\bar{T}-1,\bar{T}}(C_{\bar{T}})$ as $\gamma \to \infty$ is given by

$$\begin{aligned}
&\lim_{\gamma \to \infty}[\, - \mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1}) \cdot \\
&\frac{C^{uu}\mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{uu}} + C^{ud}\mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{ud}}}{\mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{uu}} + \mathbb{Q}^{mm}(A_{\bar{T}}B_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{ud}}} - \\
&\mathbb{Q}^{mm}(A_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})\frac{C^{du}\mathbb{Q}^{mm}(A_{\bar{T}}^c B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{du}} + C^{dd}\mathbb{Q}^{mm}(A_{\bar{T}}^c B_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{dd}}}{\mathbb{Q}^{mm}(A_{\bar{T}}^c B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{du}} + \mathbb{Q}^{mm}(A_{\bar{T}}^c B_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{dd}}}\,] \\
&= \mathbb{Q}^{mm}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\min\{C^{uu}, C^{ud}\} + \mathbb{Q}^{mm}(A_{\bar{T}}^c/\mathcal{F}_{\bar{T}-1})\min\{C^{du}, C^{dd}\} \\
&= E_{\mathbb{Q}^{mm}}[\min_{Y_{\bar{T}}} C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}].
\end{aligned} \tag{5.36}$$

In fact, $Q(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})$ is the same among all martingale measures Q equivalent to $\mathbb{P}$. Therefore, the expectation in (5.25) could be taken under any equivalent martingale measure Q ■

The theorem above shows that asymptotically as $\gamma \to 0$ and $\gamma \to \infty$ the recursive structure of the indifference price is preserved. To evaluate the benefits of continuing to hold the claim, an investor with an infinitesimally small risk aversion would price the next date's indifference value using expectation under the minimal martingale measure, while a very risk averse buyer would use the "lower hedging price" for European claims. Both investors will exercise if the value of "alternative to continue" falls below the payoff from immediate exercise C_t.

The next theorem provides another condition under which the indifference price with forward utility $a_t(C;s)$ accumulates by taking expectations under the minimal martingale measure, that is the nonlinearity in $\mathcal{A}^{t,u}_{\mathbb{Q}^{mm}}$ goes away.

Theorem 27 (Perfect correlation result) *If the historical measure $\mathbb{P}$ is such that, for all $t \leq \bar{T}$,*

$$\mathbb{P}\left(\eta_{t+1}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right), \in \{0,1\}, \tag{5.37}$$

then the forward indifference price $a_t(C;s)$ coincides with $\nu^0_t(C)$ defined by equation (5.23) above.

Proof. Under condition (5.37) $\mathbb{Q}^{mm}(Y_t/\mathcal{F}_{t-1} \vee \mathcal{F}^S_t) \in \{0,1\}$. Writing the expression for the indifference price explicitly as in (5.31), one could see that equation (5.31) for $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(a_{t+1}(C;s))$ simplifies to $E_{\mathbb{Q}^{mm}}[a_{t+1}(C;s)/\mathcal{F}_t]$, and the statement of the theorem follows ■

Condition (5.37) indicates that given all the information up to time t and a particular time $t+1$ realization of S_{t+1}, the value of Y_{t+1} is known for sure and we can say the movements of stock S and risk-factor Y are "perfectly correlated". In that case there is only one (defined on $\mathcal{F}_t$) martingale measure equivalent to $\mathbb{P}$, the nonlinearity in $a_t(C;s)$ goes away, and $a_t(C;s)$ becomes the same as $\nu^0_t(C)$.

The next result defines an early exercise time that later will be shown to be the optimal exercise time. It is an intermediate result that provide an additional characterization for the indifference price used in other theorems.

Proposition 5.1 *Let $a_t(C;s)$ be defined as in equation (5.12). Define a stopping time $\tau^\star_t$ as*

$$\tau^\star_t = \inf_{t \leq u \leq \bar{T}} \{u : a_u(C;s) = C_u\}. \tag{5.38}$$

$$a_t(C;s) = \sup_{t \le \tau \le \bar{T}} \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(a_\tau(C)) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(a_{\tau^\star_t}(C)). \tag{5.39}$$

Proof. Since the operators $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(\cdot)$ are monotone, for any $u \ge t$,

$$a_t(C;s) \ge \mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(a_{t+1}(C;s)) \ge \mathcal{E}^{t,u}_{\mathbb{Q}^{mm}}(a_u(C;s)) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(a_u(C;s)). \tag{5.40}$$

Thus, for any stopping time $t \le \tau \le \bar{T}$, $a_t(C;s) \ge \mathcal{E}^{t,\bar{T}}(a_\tau(C;s))$. Also the process $a_{u\wedge\tau^\star_t}(C;s)$, $t \le u \le \bar{T}$ satisfies

$$\mathcal{E}^{u,u+1}_{\mathbb{Q}^{mm}}(a_{u+1\wedge\tau^\star_t}(C;s)) = a_{u\wedge\tau^\star_t}(C;s). \tag{5.41}$$

Indeed, if $\tau^\star_t < u+1$ then $a_{u+1\wedge\tau^\star_t}(C;s) = a_{\tau^\star_t}(C;s)$ is $\mathcal{F}_u$-measurable and the left-hand side of (5.41) equals $a_{\tau^\star_t}(C;s)$. If $\tau^\star_t \ge u+1$ then the righthand side of (5.41) becomes $\mathcal{E}^{u,u+1}_{\mathbb{Q}^{mm}}(a_{u+1}(C;s))$, which is equal to $a_u(C;s)$ by definition of $\tau^\star_t$ (see equation (5.38)). Repeated application of (5.41) for $s = t, t+1, ..., \bar{T}$ yields $a_t(C;s) = \mathcal{E}^{t,\bar{T}}(a_{\bar{T}\wedge\tau^\star_t}(C;s)) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(a_{\tau^\star_t}(C;s))$ ■

The next result provides an alternative characterization of the indifference price. It is consistent with the Snell envelope representation in complete market. In addition, it confirms that the indifference price $a_t(C;s)$ is greater than any of the prices $\mathcal{E}^{t,s}_{\mathbb{Q}^{mm}}(C_u)$, $t \le u \le \bar{T}$, of corresponding European claims.

Theorem 28 *Define a sequence of $\mathcal{F}_t$-measurable variables ν_t as follows:*

$$\nu_t = \sup_{t \le \tau \le \bar{T}} \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(C_\tau). \tag{5.42}$$

The indifference price process $a_t(C;s)$ and the process ν_t coincide.

Proof. For any stopping time τ, $a_\tau(C) \ge C_\tau$. The operators $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}(\cdot)$ are monotone and Proposition 5.1 yields $a_t(C;s) \ge \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(a_\tau(C)) \ge \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(C_\tau)$. On the other hand, $a_{\tau^\star_t}(C;s) = C_{\tau^\star_t}$. Therefore, $a_t(C;s) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(a_{\tau^\star_t}(C)) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(C_{\tau^\star}) = \sup_{t \le \tau \le \bar{T}} \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{mm}}(C_\tau)$ ■

Theorem 29 (Optimal stopping time) *Optimal stopping time defined in equation (5.38) and $\alpha_{t+1}, ...\alpha_{\bar{T}}$ shown in equation (5.45) are a solution of stochastic* *optimization problem (5.1), that is*

$$V^C(X_t, S_t, Y_t, t; s) = E_{\mathbb{P}}[U^F_{\tau^\star_t}\left(X_{\tau^\star_t} + C_{\tau^{\star}t}, \tau^\star_t\right); s)/\mathcal{F}_t], \tag{5.43}$$

with

$$X_{\tau^\star_t} = \sum_{u=t}^{\tau^\star_t}(S_{u+1} - S_u)\alpha^\star_{u+1}, \tag{5.44}$$

$$\begin{gathered}\alpha^\star_t = \frac{1}{\gamma S_{t-1}(\xi^u - \xi^d)} \log\left(\frac{(\xi^u - 1)\mathbb{P}(A_t/\mathcal{F}_{t-1})}{(1-\xi^d)\mathbb{P}(A^c_t/\mathcal{F}_{t-1})}\right) + \frac{1}{\gamma S_{t-1}(\xi^u - \xi^d)}\cdot \\ \log\left(\frac{(e^{\gamma - a^{uu}_{t+1}}\mathbb{P}(A_t, B_t/\mathcal{F}_{t-1}) + e^{\gamma - a^{ud}_{t+1}}\mathbb{P}(A_t, B^c_t/\mathcal{F}_{t-1}))\mathbb{P}(A^c_t/\mathcal{F}_{t-1})}{(e^{\gamma - a^{du}_{t+1}}\mathbb{P}(A^c_t, B_t/\mathcal{F}_{t-1}) + e^{\gamma - a^{dd}_{t+1}}\mathbb{P}(A^c_t, B^c_t/\mathcal{F}_{t-1}))\mathbb{P}(A_t/\mathcal{F}_{t-1})}\right),\end{gathered} \tag{5.45}$$

$$\begin{aligned}
&\mathbb{P}(A_{t+1}/\mathcal{F}_t) = P\left(S_{t+1} = S^u_{t+1}/\mathcal{F}_t\right), \\
&\mathbb{P}(A^c_{t+1}/\mathcal{F}_t) = P\left(S_{t+1} = S^d_{t+1}/\mathcal{F}_t\right), \\
&\mathbb{P}(A_{t+1}B_{t+1}/\mathcal{F}_t) = P\left(S_{t+1} = S^u_{t+1}, Y_{t+1} = Y^u_{t+1}/\mathcal{F}_t\right), \\
&\mathbb{P}(A^c_{t+1}B_{t+1}/\mathcal{F}_t) = P\left(S_{t+1} = S^d_{t+1}, Y_{t+1} = Y^u_{t+1}/\mathcal{F}_t\right), \\
&\mathbb{P}(A_{t+1}B^c_{t+1}/\mathcal{F}_t) = P\left(S_{t+1} = S^u_{t+1}, Y_{t+1} = Y^d_{t+1}/\mathcal{F}_t\right), \\
&\mathbb{P}(A^c_{t+1}B^c_{t+1}/\mathcal{F}_t) = P\left(S_{t+1} = S^d_{t+1}, Y_{t+1} = Y^d_{t+1}/\mathcal{F}_t\right),
\end{aligned} \tag{5.46}$$

and a^{uu}_{t+1}, a^{ud}_{t+1}, a^{du}_{t+1} and a^{dd}_{t+1} denoting the four possible values of $a_{t+1}(C; s)$, as viewed from time t.

Proof. We first check the result for $t = \bar{T} - 1$.

$$\begin{aligned}
&E_{\mathbb{P}}[U^F_{\bar{T}-1}(X_{\tau^\star_{\bar{T}-1}} + C_{\tau^\star_{\bar{T}-1}}; s)/\mathcal{F}_{\bar{T}-1}] = \\
&E_{\mathbb{P}}[\mathbf{1}_{\{\tau^\star_{\bar{T}-1} = \bar{T}-1\}}U^F_{\tau^\star_{\bar{T}-1}}(X_{\tau^\star_{\bar{T}-1}} + C_{\tau^\star_{\bar{T}-1}}; s) + \\
&(1 - \mathbf{1}_{\{\tau^\star_{\bar{T}-1} = \bar{T}-1\}})U^F_{\tau^\star_{\bar{T}-1}}(X_{\tau^\star_{\bar{T}-1}} + C_{\tau^\star_{\bar{T}-1}}; s)/\mathcal{F}_{\bar{T}-1}] = \\
&\mathbf{1}_{\{\tau^\star_{\bar{T}-1} = \bar{T}-1\}}U^F_{\bar{T}-1}(X_{\bar{T}-1} + C_{\bar{T}-1}; s) + \\
&(1 - \mathbf{1}_{\{\tau^\star_{\bar{T}-1} = \bar{T}-1\}})E_{\mathbb{P}}[U^F_{\bar{T}}(X_{\bar{T}} + C_{\bar{T}}; s)/\mathcal{F}_{\bar{T}-1}] = \\
&\mathbf{1}_{\{\tau^\star_{\bar{T}-1} = \bar{T}-1\}}U^F_{\bar{T}-1}(X_{\bar{T}-1} + C_{\bar{T}-1}; s) + \\
&(1 - \mathbf{1}_{\{\tau^\star_{\bar{T}-1} = \bar{T}-1\}})E_{\mathbb{P}}[V^C(X_{\bar{T}}, S_{\bar{T}}, Y_{\bar{T}}, \bar{T}; s)/\mathcal{F}_{\bar{T}-1}].
\end{aligned} \tag{5.47}$$

By definition of $\tau^\star_{\bar{T}-1}$,

$$\begin{aligned}&(1-\mathbf{1}_{\{\tau^\star_{\bar{T}-1}=\bar{T}-1\}})E_{\mathbb{P}}[V^C(X_{\bar{T}},S_{\bar{T}},Y_{\bar{T}},\bar{T};s)/\mathcal{F}_{\bar{T}-1}]=\\&(1-\mathbf{1}_{\{\tau^\star_{\bar{T}-1}=\bar{T}-1\}})V^C(X_{\bar{T}-1},S_{\bar{T}-1},Y_{\bar{T}-1},\bar{T}-1;s)\end{aligned} \tag{5.48}$$

for $\alpha^\star_{\bar{T}}$ defined as in (5.45). Therefore

$$\begin{aligned}&E_{\mathbb{P}}[U^F_{\tau^\star_{\bar{T}-1}}(X_{\tau^\star_{\bar{T}-1}}+C_{\tau^\star_{\bar{T}-1}};s)/\mathcal{F}_{\bar{T}-1}]=\\&V^C(X_{\bar{T}-1},S_{\bar{T}-1},Y_{\bar{T}-1};s).\end{aligned} \tag{5.49}$$

Now assume that (5.43) holds for $t=u+1,...\bar{T}$. To show that (5.43) also holds for $t=u$, write

$$\begin{aligned}&E_{\mathbb{P}}[U^F_{\tau^\star_u}(X_{\tau^\star_u}+C_{\tau^\star_u};s)/\mathcal{F}_u]=E_{\mathbb{P}}[\mathbf{1}_{\{\tau^\star_u=u\}}U^F_{\tau^\star_u}(X_{\tau^\star_u}+C_{\tau^\star_u};s)+\\&(1-\mathbf{1}_{\{\tau^\star_u=u\}})U^F_{\tau^\star_u}(X_{\tau^\star_u}+C_{\tau^\star_u};s)/\mathcal{F}_u]=\mathbf{1}_{\{\tau^\star_u=u\}}U^F_u(X_u+C_u;s)+\\&(1-\mathbf{1}_{\{\tau^\star_u=u\}})E_{\mathbb{P}}[U^F_{\tau^\star_u}(X_{\tau^\star_u}+C_{\tau^\star_u};s)/\mathcal{F}_u].\end{aligned} \tag{5.50}$$

For $\{w:\tau^\star_u(w)>u\}$, $\tau^\star_u=\tau^\star_{u+1}$ and

$$\begin{aligned}&E_{\mathbb{P}}[U^F_{\tau^\star_u}(X_{\tau^\star_u}+C_{\tau^\star_u};s)/\mathcal{F}_u]=\mathbf{1}_{\{\tau^\star_u=u\}}V^C(X_u,S_u,Y_u,u;s)+\\&(1-\mathbf{1}_{\{\tau^\star_u=u\}})E_{\mathbb{P}}[U^F(X_{\tau^\star_{u+1}}+C_{\tau^\star_{u+1}},\tau^\star_{u+1},u+1;s)/\mathcal{F}_u]=\\&\mathbf{1}_{\{\tau^\star_u=u\}}V^C(X_u,S_u,Y_u,u;s)+(1-\mathbf{1}_{\{\tau^\star_u=u\}})E_{\mathbb{P}}[V^C(X_{u+1},u+1,u+1;s)/\mathcal{F}_u]\end{aligned} \tag{5.51}$$

by induction assumption. By definition of $\tau^\star_u$,

$$\begin{aligned}&(1-\mathbf{1}_{\{\tau^\star_u=u\}})E_{\mathbb{P}}[V^C(X_{u+1},S_{u+1},Y_{u+1},u+1,u+1;s)/\mathcal{F}_u]=\\&(1-\mathbf{1}_{\{\tau^\star_u=u\}})V^C(X_u,S_u,Y_u,u;s)\end{aligned} \tag{5.52}$$

for $\alpha^\star_{u+1}$ defined as in (5.45).
Therefore, $E_{\mathbb{P}}[U^F_{\tau^\star_u}(X_{\tau^\star_u}+C_{\tau^\star_u};s)/\mathcal{F}_u]=V^C(X_u,S_u,Y_u,u;s)$ ■

5.2 Dynamic indifference valuation with the backward utility.

In this section we show how liabilities with early exercise CAN be valued in the discrete indifference framework using the backward dynamic utility. While the forward dynamic utility introduced in the previous section is a new concept, the backward dynamic utility is closer to the classical static exponential utility. In fact, the backward utility is the unique self-generating dynamic extension of the classical static exponential utility. We have already presented the valuation for forward and backward utilities for European claims. The forward valuation approach has a number of advantages, such as independence of the investment horizon and simpler characterization of the measure, but the backward utility approach possesses all other natural properties of the forward one.

We formulate the corresponding value functions for the agent as stochastic control problems incorporating both the agent's investment policy and the exercise time of the claim. We define and derive closed form formulas for the indifference price. We investigate properties of the indifference price with respect to risk aversion, in particular its monotonicity and limiting values as risk aversion becomes infinitely small or infinitely large. We provide a characterization of optimal exercise time and optimal hedging policy, and derive a new representation for the early exercise indifference price that relates it to the indifference prices of European claims of maturities within the allowed exercise horizon. We comment on how the results for the backward dynamic utility compare to tse for the forward utility. Numerical examples for both forward and backward utilities are provided in chapter 8. In this chapter we discuss valuation exclusively from the buyer's perspective, and all the nonlinear operators and functions used to characterize the indifference price refer to the buyer's point of view.

Let t be the current valuation time, $\alpha = (\alpha_{t+1}, ..., \alpha_{\bar{T}})$ be a trading strategy and τ be a stopping time (to exercise the claim) chosen by the investor. Here we only consider early exercise on a finite horizon and thus we require $t \le \tau \le \bar{T}$ a.s. We also require that the claim in exercised at $\bar{T}$ if it has not been exercise before that time. The early exercise liability yields intrinsic payoff C_τ if exercised at time τ. The total wealth at time τ of the investor combines proceeds X_τ accumulated through trading with the payoff upon exercise C_τ. At time τ, the agent measures his total wealth through the backward utility function $U^B_\tau(X_\tau + C_\tau; T)$, normalized at T, $\bar{t} \le T$. The backward utility process U^B_t is the same one used in [39] for European contracts. Investor's time t value function is defined accordingly:

Definition 5.3 *Let T be the backward normalization point. The time t value function of the buyer of the American claim C, written at t_0 and expiring at $\bar{T}$, is defined* *as the expected utility of his total wealth upon exercise, optimized through the choice of a trading strategy α and of a stopping time τ. Namely,*

$$V^C(X_t, S_t, Y_t, t; T) = \sup_{\alpha_{t+1},\dots,\alpha_{\bar{T}}} \sup_{t \le \tau \le \bar{T}} E_{\mathbb{P}}[U^B_\tau(X_\tau + C_\tau; T)/\mathcal{F}_t]. \tag{5.53}$$

Given the notion of the buyer's value function, we define the indifference price as:

Definition 5.4 *The buyer's indifference price of the American claim C, written at t_0 and expiring at $\bar{T} \le T$, is the amount $a_t(C;T)$, to be payed for the claim, which makes the value function V^C equal to U^B. Namely, $a_t(C;T)$ is the amount satisfying:*

$$V^C(X_t, S_t, Y_t, t; T) = U^B_t(X_t + a_t(C;T), T). \tag{5.54}$$

We aim to constructing the recursive algorithm that would relate the time t indifference price to the one of time $t+1$. Indifference prices are obtained by studying the corresponding value functions. Thus, one should proceed by relating the time t buyer's value function to that of time $t+1$. In the discrete time model, the claim can either be exercised immediately, or the investor will have to wait till the next period to be able to exercise the claim. Using this observation we arrive at a simplified expression for the value function.

Theorem 30 *If the optimal $\tau^\star_u$ and $\alpha^\star_{u+1}, \dots, \alpha^\star_{\bar{T}}$ solving stochastic control problem (5.53) exist for all $t \le u \le T$, then $V^C(X_t, S_t, Y_t, t; T)$ has the following recursive representation:*

$$\begin{cases} V^C(X_t, S_t, Y_t, t; T) = \\ \max\left\{U^B_t(X_t + C_t; T), \sup\limits_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t]\right\}, \\ V^C(X_{\bar{T}}, S_{\bar{T}}, Y_{\bar{T}}, \bar{T}; T) = U^B_{\bar{T}}(X_{\bar{T}} + C_{\bar{T}}; T). \end{cases} \tag{5.55}$$

Proof. If optimizers $\tau^\star_u$ and $\alpha^\star$ exist for all $t \le u \le \bar{T}$, then

$$V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T) = E_{\mathbb{P}}[U^B_{\tau^\star_{t+1}}(X_{\tau^\star_{t+1}} + C_{\tau^\star_{t+1}}; T)/\mathcal{F}_{t+1}]. \tag{5.56}$$

$\tau^\star_{t+1}$ and $\alpha^\star_{t+2}, \dots, \alpha^\star_{\bar{T}}$ may not be optimal solutions of (5.53) at time t, implying for any choice of α_{t+1} that $V^C(X_t, S_t, Y_t, t; T) \ge E_{\mathbb{P}}[U^B_{\tau^\star_{t+1}}(X_{\tau^\star_{t+1}} + C_{\tau^\star_{t+1}}; T)/\mathcal{F}_t]$ for

$X_{\tau^\star_{t+1}} = X_t + \alpha_{t+1}(S_{t+1} - S_t) + \sum_{u=t+2}^{\tau^\star_{t+1}} \alpha^\star_u (S_u - S_{u-1})$. Conditioning on $\mathcal{F}_{t+1}$, we get

$$\begin{aligned} &V^C(X_t, S_t, Y_t, t; T) \geq E_{\mathbb{P}}[E_{\mathbb{P}}[U^B_{\tau^\star_{t+1}}(X_{\tau^\star_{t+1}} + C_{\tau^\star_{t+1}}; T)/\mathcal{F}_{t+1}]/\mathcal{F}_t] = \\ &E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t], \end{aligned} \tag{5.57}$$

for $X_{t+1} = X_t + \alpha_{t+1}(S_{t+1} - S_t)$. Since α_{t+1} is arbitrary,

$$V^C(X_t, S_t, Y_t, t; T) \geq \sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t]. \tag{5.58}$$

Defintion 5.3 clearly implies $V^C(X_t, S_t, Y_t, t; T) \geq U^B_t(X_t + C_t; T)$. Taking this into account together with (5.58), we conclude $V^C(X_t, S_t, Y_t, t; T)$ is at least as big as the righthand side of (5.55). On the other hand, for any τ and α,

$$\begin{aligned} &E_{\mathbb{P}}[U^B_\tau(X_\tau + C_\tau; T)/\mathcal{F}_t] = \\ &\mathbf{1}_{\{\tau=t\}} U^B_t(X_t + C_t; T) + (1 - \mathbf{1}_{\{\tau=t\}}) E_{\mathbb{P}}[U^B_\tau(X_\tau + C_\tau; T)/\mathcal{F}_t] = \\ &\mathbf{1}_{\{\tau=t\}} U^B_t(X_t + C_t; T) + (1 - \mathbf{1}_{\{\tau=t\}}) E_{\mathbb{P}}[E_{\mathbb{P}}[U^B_\tau(X_\tau + C_\tau; T)/\mathcal{F}_{t+1}]/\mathcal{F}_t] \leq \\ &\mathbf{1}_{\{\tau=t\}} U^B_t(X_t + C_t; T) + (1 - \mathbf{1}_{\{\tau=t\}}) E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t] \leq \\ &\mathbf{1}_{\{\tau=t\}} U^B_t(X_t + C_t; T) + (1 - \mathbf{1}_{\{\tau=t\}}) \sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t] \leq \\ &\mathbf{1}_{\{\tau=t\}} \max\left\{ U^B_t(X_t + C_t; T), \sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t] \right\} + \\ &(1 - \mathbf{1}_{\{\tau=t\}}) \max\left\{ U^B_t(X_t + C_t; T), \sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t] \right\} = \\ &\max\left\{ U^B_t(X_t + C_t; T), \sup_{\alpha_{t+1}} E_{\mathbb{P}}[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1; T)/\mathcal{F}_t] \right\} \end{aligned} \tag{5.59}$$

■ To characterize the buyer's indifference price of the American claim C, we use a family of operators $\mathcal{A}^{t,u}_Q(Z)$ for $t \leq u \leq \bar{T}$. We let Z be an $\mathcal{F}_u$ - measurable random variable and Q be a martingale measure equivalent to $\mathbb{P}$. We define $\mathcal{A}^{t,u}_Q(Z)$ as

$$\begin{cases} \mathcal{A}^{t,u}_Q(Z) = \mathcal{A}^{t,u-1}_Q\left(\mathcal{A}^{u-1,u}_Q(Z)\right), \\ \mathcal{A}^{t,t}_Q(Z) = Z, \end{cases} \tag{5.60}$$

where

$$\mathcal{A}^{u-1,u}_Q(Z) = \max\left\{ C_{u-1}, \mathcal{E}^{u-1,u}_Q(Z) \right\}. \tag{5.61}$$

with $\mathcal{E}_Q^{u-1,u}(Z)$ is the one-period European buyer's operator, defined in chapter 2 equation (2.9). The structural form of nonlinear operators defined above is the same we used for the forward indifference price in equation (5.9). The theorem below indicates the measure that should be used in connection with the above operators to value American contracts. The theorem below is the multi-period pricing algorithm, the main result of this section.

Theorem 31 *Let $\mathbb{Q}^{me}$ be the minimal entropy martingale measure, defined in chapter 3.*
(i) The indifference price $a_t(C;T)$ as in definition (5.4) satisfies

$$\begin{cases} a_t(C;T) = \max\left\{C_t, \mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}\left(a_{t+1}(C;T)\right)\right\}, t < \bar{T} \\ a_{\bar{T}}(C;T) = C_{\bar{T}}, \end{cases} \tag{5.62}$$

with $\mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}$ defined in equation (2.9) of chapter 2 for $Q = \mathbb{Q}^{me}$.
(ii) The indifference price process is given by

$$a_t(C;T) = \mathcal{A}_{\mathbb{Q}^{me}}^{t,\bar{T}}\left(C_{\bar{T}}\right), \tag{5.63}$$

with $\mathcal{A}_{\mathbb{Q}^{me}}^{t,u}$ defined in equations (5.60)-(5.61) for $Q = \mathbb{Q}^{me}$ and $t \le u \le \bar{T}$.
(iii) The pricing algorithm is consistent across time, in that, for $0 \le t \le u \le \bar{T}$, the semi-group property

$$a_t(C;T) = \mathcal{A}_{\mathbb{Q}^{me}}^{t,u}\left(\mathcal{A}_{\mathbb{Q}^{me}}^{u,\bar{T}}\left(C_{\bar{T}}\right)\right) = \mathcal{A}_{\mathbb{Q}^{me}}^{t,u}\left(a_u(C;T)\right) = a_t\left(\mathcal{A}_{\mathbb{Q}^{me}}^{u,\bar{T}}\left(C_{\bar{T}}\right);T\right) \tag{5.64}$$

holds.

Formula (5.62) has an intuitive interpretation: C_t equals the amount the buyer gains if he exercises the claim immediately; $\mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}\left(a_{t+1}(C;T)\right)$ represents the amount the buyer gains if he continues to hold the claim till the next possible exercise time. $a_t(C;T)$ is then the maximum of the two available alternatives. Formula (5.62) inherits the nested structure of the discrete time complete market no-arbitrage prices, rolling backwards from the terminal time $\bar{T}$ to the current time t. As in the complete market case, the price is the maximum of two values, one of which is the intrinsic value of the option, and the other is the value of the alternative "to continue".

Although it is similar in the structural form to the complete market price, our pricing scheme reflects market incompleteness in a number of ways. One is in the choice of the pricing measure, which differs from the complete market case. The measure $\mathbb{Q}^{me}$ used throughout is the minimal entropy martingale measure, the same measure as used in indifference pricing of European claims with backward preferences. Another consequence of market incompleteness is that the value of alternative "to continue" is now characterized using the nonlinear one-period European buyer's indifference prices, and not the risk-neutral expectations. The nonlinearity enters though the level of the absolute risk aversion. As we see further, the nonlinearity goes away in the limiting cases of risk aversion approaching zero, or in the case of "perfect correlation" discussed below.

Remarkably, the backward early exercise price only differs from the forward early exercise price in the choice of the pricing measure (the minimal entropy one and not the minimal martingale). The definition of operators $\mathcal{A}_Q^{t,u}(\cdot)$ does not contain any dependence on the backward normalization point T. One may wrongly conclude that the prices may be independent on the normalization point as well. This is, in fact, not the case since the characterization of the martingale measure $\mathbb{Q}^{me}$ itself includes dependence of the backward normalization point T. Through the measure $\mathbb{Q}^{me}$, the prices become dependent on the backward normalization point T.

Proof. (i) At time $\bar{T}-1$,

$$
\begin{aligned}
&V^C(X_{\bar{T}-1}, S_{\bar{T}-1}, Y_{\bar{T}-1}, \bar{T}-1; T) = \\
&\max\left\{U^B_{\bar{T}-1}(X_{\bar{T}-1}+C_{\bar{T}-1};T), \sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}\left[V^C(X_{\bar{T}}, S_{\bar{T}}, Y_{\bar{T}}, \bar{T}; T)/\mathcal{F}_{\bar{T}-1}\right]\right\} = \\
&\max\left\{U^B_{\bar{T}-1}(X_{\bar{T}-1}+C_{\bar{T}-1};T), \sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}\left[U^B_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}};T)/\mathcal{F}_{\bar{T}-1}\right]\right\}.
\end{aligned} \tag{5.65}
$$

$\sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}\left[U^B_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}};T)/\mathcal{F}_{\bar{T}-1}\right]$ can be viewed as time $\bar{T}-1$ value function of an investor with forward preferences holding a claim $C_{\bar{T}}$ expiring at $\bar{T}$. Considering the results already obtained for European claims,

$$
\sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}\left[U^B_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}};T)/\mathcal{F}_{\bar{T}-1}\right] = U^B_{\bar{T}}(X_{\bar{T}}+\mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{me}}(C_{\bar{T}});T), \tag{5.66}
$$

and

$$
\begin{aligned}
&V^C(X_{\bar{T}-1}, S_{\bar{T}-1}, Y_{\bar{T}-1}, \bar{T}-1;T) = \\
&\max\left\{U^B_{\bar{T}-1}(X_{\bar{T}-1}+C_{\bar{T}-1};T), U^B_{\bar{T}-1}(X_{\bar{T}} - \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{me}}\left(C_{\bar{T}}\right);T)\right\} = \\
&U^B_{\bar{T}-1}\left(X_{\bar{T}-1} + \max\left\{C_{\bar{T}-1}, \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{me}}\left(C_{\bar{T}}\right)\right\};T\right).
\end{aligned}
\tag{5.67}
$$

This implies the corresponding relation for the indifference price. Following a typical induction argument, assume that the formula holds for times $t+1$ through $\bar{T}$. Now we need to prove that the formula also holds at time t. From definition 5.4 of the indifference price,

$$
\begin{aligned}
&\sup_{\alpha_{t+1}} E_{\mathbb{P}}\left[V^C(X_{t+1}, S_{t+1}, Y_{t+1}, t+1;T)/\mathcal{F}_t\right] = \\
&\sup_{\alpha_{t+1}} E_{\mathbb{P}}\left[U^B_{t+1}(X_{t+1}+a_{t+1}(C;T);T)/\mathcal{F}_t\right] = \\
&\sup_{\alpha_{t+1}} E\left[-\exp\left(-\gamma(X_{t+1}+a_{t+1}(C;T)+\frac{1}{\gamma}\mathcal{H}(t+1;T))\right)/\mathcal{F}_t\right] = \\
&-\exp\left(-\gamma\left(X_t + \mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}\left(a_{t+1}(C;T)+\frac{1}{\gamma}\mathcal{H}(t+1;T)\right)\right) - h_{t+1}\right) = \\
&U^B_t\left(X_t + \mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}\left(a_{t+1}(C;T)+\frac{1}{\gamma}\mathcal{H}(t+1;T)\right) + \frac{1}{\gamma}h_{t+1} - \frac{1}{\gamma}\mathcal{H}(t;T);T\right) = \\
&U^B_t\left(X_t + \mathcal{E}^{t,t+1}_{\mathbb{Q}^{mm}}\left(a_{t+1}(C;T)+\frac{1}{\gamma}\mathcal{H}(t+1;T)\right)\right. \\
&\left.+\frac{1}{\gamma}h_{t+1} - \left(\frac{1}{\gamma}h_{t+1} + \mathcal{E}^{t,T}_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}(t+1;T)\right)\right);T\right) = \\
&U^B_t\left(X_t + \mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}\left(a_{t+1}(C;T)\right);T\right).
\end{aligned}
\tag{5.68}
$$

Therefore, using formula (5.55) for the buyer's value function,

$$
\begin{aligned}
&V^C(X_t, S_t, Y_t, t;T) = \max\left\{U^B_t(X_t+C_t;T), U^B_t]\left(X_t + \mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}\left(a_{t+1}(C;T)\right);T\right)\right\} = \\
&U^B_t\left(X_t + \max\left\{C_t, \mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}\left(a_{t+1}(C;T)\right)\right\};T\right).
\end{aligned}
\tag{5.69}
$$

From the definition of the buyer's indifference price and the calculation above,

$$
a_t\left(C;T\right) = \max\left\{C_t, \mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}\left(a_{t+1}\left(C;T\right)\right)\right\}, \tag{5.70}
$$

and the proof of Part (i) is complete.

Parts (ii) and (iii) of theorem 31 follow from (i) together with the definition of the operators $\mathcal{A}^{t,u}_{\mathbb{Q}^{me}}$ introduced above in equations (5.60) and (5.61) ■ We further consider properties of the buyer's indifference price. The next theorem (32) shows how the price is affected by changes in the absolute risk aversion. Part (i) of the theorem shows that a more risk averse buyer assigns a smaller value to the claim bearing unhedgeable risk. There, we use a slightly different notation $a_t(C\,;\gamma)$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}(Z_{t+1}\,;\gamma)$. The "extra argument" γ specifies a particular level of risk aversion to which the indifference price corresponds. Part (ii) of the theorem characterizes asymptotic behavior as $\gamma \to 0$. Also, it provides an upper bound for $a_t(C\,;\gamma)$ for all values of γ, obtained as the indifference price of a buyer with infinitesimally small risk aversion. The limiting value shows significant structural similarity to the no-arbitrage price of an American claim in a complete market. However it uses a different measure $\mathbb{Q}^{me}$, and not the classical risk-neutral measure. Results of Part (ii) are consistent with familiar asymptotic results for European derivatives in discrete and continuous time. Part (iii) describes the asymptotic behavior of the price as $\gamma \to \infty$ and provides a lower bound for $a_t(C;T)$ in the form of the buyer's indifference price for an agent with infinitely large risk aversion.

Theorem 32 (Monotonicity and asymptotic bounds) *(i) For $\gamma_1 \geq \gamma_2$*

$$a_t(C;\gamma_1) \leq a_t(C;\gamma_2)\,,\ \ t \leq \bar{T}\ \ a.s. \tag{5.71}$$

(ii) Define the sequence of $\mathcal{F}_t$-measurable random variables $\nu_t^0(C)$ as

$$\begin{cases} \nu_{\bar{T}}^0(C) = C_{\bar{T}}, \\ \nu_t^0(C) = \max\left\{C_t, E_{\mathbb{Q}^{me}}\left[\nu_{t+1}^0(C)/\mathcal{F}_t\right]\right\},\ \ t < \bar{T}\ . \end{cases} \tag{5.72}$$

Then for any $t \leq \bar{T}$,

$$a_t(C;\gamma) \nearrow \nu_t^0(C)\,,\ \ as\ \gamma \to 0. \tag{5.73}$$

(iii) Define the sequence of $\mathcal{F}_t$-measurable random variables $\nu_t^{inf}(C)$ as

$$\begin{cases} \nu_{\bar{T}}^{inf}(C) = C_{\bar{T}}, \\ \nu_t^{inf}(C;\gamma) = \max\left\{C_t, E_{\mathbb{Q}^{me}}[\min\limits_{Y_{t+1}} \nu_{t+1}^{inf}/\mathcal{F}_t]\right\}, t < T. \end{cases} \tag{5.74}$$

Then

$$a_t(C;T) \searrow \nu_t^{inf}(C),\ \ as\ \gamma \to \infty. \tag{5.75}$$

In (5.74), the expectation can be taken under any equivalent to $\mathbb{P}$ martingale measure

Q.

Proof. (i) For $t = T$, $a_{\bar{T}}(C;\gamma_1) = a_{\bar{T}}(C;\gamma_2) = C_{\bar{T}}$ and inequality (5.71) holds. For $t < \bar{T}$, one only needs to show that, for every $\mathcal{F}_{t+1}$-measurable random variable Z_{t+1},

$$\mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(Z_{t+1};\gamma_1) \leq \mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(Z_{t+1};\gamma_2). \tag{5.76}$$

Let $\tilde{\gamma} = -\gamma$,

$$\begin{aligned}&\mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(Z_{t+1};\gamma) = E_{\mathbb{Q}^{me}}\left[\ln\left(E_{\mathbb{Q}^{me}}\left[\left(e^{Z_{t+1}}\right)^{\tilde{\gamma}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]\right)^{\frac{1}{\tilde{\gamma}}}/\mathcal{F}_t\right] = \\ &E_{\mathbb{Q}^{me}}\left[\ln\left(E_{\mathbb{Q}^{me}}\left[\left(Z1_{t+1}\right)^{\tilde{\gamma}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]\right)^{\frac{1}{\tilde{\gamma}}}/\mathcal{F}_t\right]\end{aligned} \tag{5.77}$$

where $Z1_{t+1} = e^{Z_{t+1}} > 0$. Due to the monotonicity of the conditional expectation and logarithmic function it only remains to show that for $\tilde{\gamma_1} \leq \tilde{\gamma_2} < 0$

$$E_{\mathbb{Q}^{me}}\left[\left(Z1_{t+1}\right)^{\tilde{\gamma_1}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]^{\frac{1}{\tilde{\gamma_1}}} \leq E_{\mathbb{Q}^{me}}\left[\left(Z1_{t+1}\right)^{\tilde{\gamma_2}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]^{\frac{1}{\tilde{\gamma_2}}} \tag{5.78}$$

With $Z1^{\tilde{\gamma_1}}_{t+1} = G_{t+1}$ and $0 < \hat{\gamma} = \frac{\tilde{\gamma_2}}{\tilde{\gamma_1}} \leq 1$, inequality (5.78) can be rewritten (note that the sign of the new inequality is different because $\frac{1}{\tilde{\gamma_1}}$ is negative):

$$E_{\mathbb{Q}^{me}}\left[G_{t+1}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right] \geq E_{\mathbb{Q}^{me}}\left[\left(G_{t+1}\right)^{\hat{\gamma}}/\mathcal{F}_t \vee \mathcal{F}^S_{t+1}\right]^{\frac{1}{\hat{\gamma}}} \tag{5.79}$$

The last inequality follows from concavity of the function $x^{\hat{\gamma}}$ for $x > 0$, $0 < \hat{\gamma} \leq 1$, and the Jensen's inequality.
(ii) We first show that the formula holds for time $\bar{T} - 1$. For any t, including $t = \bar{T} - 1$, $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(C_{\bar{T}})$ can be written explicitly as

$$\begin{aligned}&\mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(a_{t+1}) = - \\ &\frac{\mathbb{Q}^{me}(A_{t+1}/\mathcal{F}_t)}{\gamma}\log\left(\frac{\mathbb{Q}^{me}(A_{t+1}B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A_{t+1}/\mathcal{F}_t)}e^{-\gamma a^{uu}_{t+1}} + \frac{\mathbb{Q}^{me}(A_{t+1}B^c_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A_{t+1}/\mathcal{F}_t)}e^{-\gamma a^{ud}_{t+1}}\right) - \\ &\frac{\mathbb{Q}^{me}(A^c_{t+1}/\mathcal{F}_t)}{\gamma}\log\left(\frac{\mathbb{Q}^{me}(A^c_{t+1}B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A^c_{t+1}/\mathcal{F}_t)}e^{-\gamma a^{du}_{t+1}} + \frac{\mathbb{Q}^{me}(A^c_{t+1}B^c_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A^c_{t+1}/\mathcal{F}_t)}e^{-\gamma a^{dd}_{t+1}}\right),\end{aligned} \tag{5.80}$$

with

$$
\begin{aligned}
&\mathbb{Q}^{me}(A_{t+1}/\mathcal{F}_t) = \mathbb{Q}^{me}\left(S_{t+1} = S^u_{t+1}/\mathcal{F}_t\right),\\
&\mathbb{Q}^{me}(A^c_{t+1}/\mathcal{F}_t) = \mathbb{Q}^{me}\left(S_{t+1} = S^d_{t+1}/\mathcal{F}_t\right),\\
&\mathbb{Q}^{me}(A_{t+1}B_{t+1}/\mathcal{F}_t) = \mathbb{Q}^{me}\left(S_{t+1} = S^u_{t+1}, Y_{t+1} = Y^u_{t+1}/\mathcal{F}_t\right),\\
&\mathbb{Q}^{me}(A^c_{t+1}B_{t+1}/\mathcal{F}_t) = \mathbb{Q}^{me}\left(S_{t+1} = S^d_{t+1}, Y_{t+1} = Y^u_{t+1}/\mathcal{F}_t\right),\\
&\mathbb{Q}^{me}(A_{t+1}B^c_{t+1}/\mathcal{F}_t) = \mathbb{Q}^{me}\left(S_{t+1} = S^u_{t+1}, Y_{t+1} = Y^d_{t+1}/\mathcal{F}_t\right),\\
&\mathbb{Q}^{me}(A^c_{t+1}B^c_{t+1}/\mathcal{F}_t) = \mathbb{Q}^{me}\left(S_{t+1} = S^d_{t+1}, Y_{t+1} = Y^d_{t+1}/\mathcal{F}_t\right),
\end{aligned}
\tag{5.81}
$$

with a^{uu}_{t+1}, a^{ud}_{t+1}, a^{du}_{t+1} and a^{dd}_{t+1} denoting the four possible values of $a_{t+1}(C;\gamma)$, as viewed from time t. For $t = \bar{T} - 1$ in particular,

$$
a^{ij}_{\bar{T}} = C^{ij} = C(S^i_{\bar{T}}, Y^j_{\bar{T}}),\ \text{for } i,j = \text{'}u\text{' and '}d\text{'}\ . \tag{5.82}
$$

Using L'Hopital's rule and equation (5.80), the limit of $\mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{me}}(C_{\bar{T}})$ as $\gamma \to 0$ is

$$
\begin{aligned}
&\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\left(\frac{\mathbb{Q}^{me}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}C^{uu} + \frac{\mathbb{Q}^{me}(A_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}C^{ud}\right)\\
&+\mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\left(\frac{\mathbb{Q}^{me}(A^c_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}C^{du} + \frac{\mathbb{Q}^{me}(A^c_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}{\mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})}C^{dd}\right)\\
&= E_{\mathbb{Q}^{me}}\left[C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}\right].
\end{aligned}
\tag{5.83}
$$

The calculation above shows that $\mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{me}}(C_{\bar{T}})$ converges to $E_{\mathbb{Q}^{me}}\left[C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}\right]$. Consequently, since $C_{\bar{T}-1}$ is independent of γ,

$$
a_{\bar{T}-1}(C;\gamma) \to \max\{C_{\bar{T}-1}, E_{\mathbb{Q}^{me}}\left[C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}\right]\},
$$

as $\gamma \to 0$.

To confirm the formula for $t \leq \bar{T} - 2$, one would proceed by induction and assume that the limit of $a_u(C;\gamma)$ as $\gamma \to 0$ equals $\nu^0_u(C)$ for all $u \geq t+1$. $a_t(C;\gamma)$ is the maximum of C_t and $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(a_{t+1}(C;\gamma))$. The latter has an explicit representation provided in equation (5.80). Since, by induction assumption, $a_{t+1}(C;\gamma)$ converges to $\nu^0_{t+1}(C)$ as $\gamma \to 0$, applying the same argument as before yields

$$
\begin{aligned}
&\lim_{\gamma\to 0} \mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(a_{t+1}(C;\gamma)) = \\
&\mathbb{Q}^{me}(A_{t+1}/\mathcal{F}_t)\left(\frac{\mathbb{Q}^{me}(A_{t+1}B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A_{t+1}/\mathcal{F}_t)}a^{uu}_{t+1} + \frac{\mathbb{Q}^{me}(A_{t+1}B^c_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A_{t+1}/\mathcal{F}_t)}a^{ud}_{t+1}\right) + \\
&\mathbb{Q}^{me}(A^c_{t+1}/\mathcal{F}_t)\left(\frac{\mathbb{Q}^{me}(A^c_{t+1}B_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A^c_{t+1}/\mathcal{F}_t)}a^{du}_{t+1} + \frac{\mathbb{Q}^{me}(A^c_{t+1}B^c_{t+1}/\mathcal{F}_t)}{\mathbb{Q}^{me}(A^c_{t+1}/\mathcal{F}_t)}a^{dd}_{t+1}\right) = \\
&E_{\mathbb{Q}^{me}}\left[\nu^0_{t+1}(C)/\mathcal{F}_t\right].
\end{aligned}
\tag{5.84}
$$

Thus $a_t(C;\gamma)$ converges to $\max\{C_t, E_{\mathbb{Q}^{me}}[\nu^0_{t+1}(C)/\mathcal{F}_t]\}$.
(iii) As before we show that formula for $t = \bar{T}-1$. For other values of $t \leq \bar{T}-2$ one could use induction and repeat the argument. The limit of $\mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}^{me}}(C_{\bar{T}})$ as $\gamma \to \infty$ is given by

$$
\begin{aligned}
&\lim_{\gamma\to\infty}\Big[-\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\cdot \\
&\frac{C^{uu}\mathbb{Q}^{me}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{uu}} + C^{ud}\mathbb{Q}^{me}(A_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{ud}}}{\mathbb{Q}^{me}(A_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{uu}} + \mathbb{Q}^{me}(A_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{ud}}} - \\
&\mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\frac{C^{du}\mathbb{Q}^{me}(A^c_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{du}} + C^{dd}\mathbb{Q}^{me}(A^c_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{dd}}}{\mathbb{Q}^{me}(A^c_{\bar{T}}B_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{du}} + \mathbb{Q}^{me}(A^c_{\bar{T}}B^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})e^{-\gamma C^{dd}}}\Big] \\
&= \mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\min\{C^{uu}, C^{ud}\} + \mathbb{Q}^{me}(A^c_{\bar{T}}/\mathcal{F}_{\bar{T}-1})\min\{C^{du}, C^{dd}\} \\
&= E_{\mathbb{Q}^{me}}[\min_{Y_{\bar{T}}} C_{\bar{T}}/\mathcal{F}_{\bar{T}-1}].
\end{aligned}
\tag{5.85}
$$

In fact, $\mathbb{Q}^{me}(A_{\bar{T}}/\mathcal{F}_{\bar{T}-1})$ is the same among all martingale measures Q equivalent to $\mathbb{P}$. Therefore, the expectation in (5.74) could be taken under any martingale measure Q ■

The theorem above shows that asymptotically as $\gamma \to 0$ and $\gamma \to \infty$ the recursive structure of the indifference price is preserved. To evaluate the benefits of continuing to hold the claim, an investor with an infinitesimally small risk aversion would price the next date's indifference value using expectation under the minimal martingale measure, while a very risk averse buyer would use the "lower hedging price" for European claims. Both investors will exercise if the value of "alternative to continue" falls below the payoff from immediate exercise C_t.

Next we present another condition, under which the backward early exercise price coincides with ν_t^0.

Theorem 33 (Perfect correlation result) *If the historical measure $\mathbb{P}$ is such that, for all $t \leq T$,*

$$\mathbb{P}\left(\eta_{t+1}/\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right) \in \{0,1\}, \tag{5.86}$$

then early exercise indifference price $a_t(C;T)$ coincides with ν_t^0 defined by equation (5.72).

Proof. Writing the expression for the indifference price explicitly as in (5.81) and taking into account the characterization of the minimal entropy martingale measure $\mathbb{Q}^{me}$, given in chapter 3 proposition 4, one can immediately see that under condition (5.86), $Q^{me}(Y_t/\mathcal{F}_{t-1} \vee \mathcal{F}_t^S) \in \{0,1\}$. Thus, equation (5.80) for $\mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}(a_{t+1}(C))$ simplifies to $E_{\mathbb{Q}^{me}}[a_{t+1}(C)/\mathcal{F}_t]$, and the statement of the theorem follows ■

Condition (5.86) indicates that, given all the information up to time t and a particular time $t+1$ realization of S_{t+1}, the value of Y_{t+1} is known for sure and we can say the movements of stock S and risk-factor Y are "perfectly correlated".

The next result defines an early exercise time that later will be shown to be the optimal exercise time. It is an intermediate result that provide an additional characterization for the indifference price used in other theorems.

Proposition 5.2 *Let $a_t(C;T)$ be defined as in equation (5.62). Define a stopping time $\tau_t^\star$ as*

$$\tau_t^\star = \inf_{t \leq s \leq \bar{T}} \{s : a_s(C;T) = C_s\}. \tag{5.87}$$

$$a_t(C;T) = \sup_{t \leq \tau \leq \bar{T}} \mathcal{E}_{\mathbb{Q}^{me}}^{t,\bar{T}}(a_\tau(C;T)) = \mathcal{E}_{\mathbb{Q}^{me}}^{t,\bar{T}}(a_{\tau_t^\star}(C;T)). \tag{5.88}$$

Proof. Since the operators $\mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}(\cdot)$ are monotone, for any $s \geq t$,

$$a_t(C;T) \geq \mathcal{E}_{\mathbb{Q}^{me}}^{t,t+1}(a_{t+1}(C)) \geq \mathcal{E}_{\mathbb{Q}^{me}}^{t,s}(a_s(C;T)) = \mathcal{E}_{\mathbb{Q}^{me}}^{t,\bar{T}}(a_s(C;T)). \tag{5.89}$$

Thus, for any stopping time $t \leq \tau \leq \bar{T}$, $a_t(C;T) \geq \mathcal{E}^{t,\bar{T}}(a_\tau(C))$. Also the process $a_{s\wedge\tau_t^\star}(C)$, $t \leq s \leq \bar{T}$ satisfies

$$\mathcal{E}^{s,s+1}_{\mathbb{Q}^{me}}(a_{s+1\wedge\tau_t^\star}(C)) = a_{s\wedge\tau_t^\star}(C). \tag{5.90}$$

Indeed, if $\tau_t^\star < s+1$ then $a_{s+1\wedge\tau_t^\star}(C) = a_{\tau_t^\star}(C)$ is $\mathcal{F}_s$-measurable and the left-hand side of (5.90) equals $a_{\tau_t^\star}(C)$. If $\tau_t^\star \geq s+1$ then the righthand side of (5.90) becomes $\mathcal{E}^{s,s+1}_{\mathbb{Q}^{me}}(a_{s+1}(C))$, which is equal to $a_s(C;T)$ by definition of $\tau_t^\star$ (see equation (5.87)). Repeated application of (5.90) for $s = t, t+1, ..., \bar{T}$ yields $a_t(C;T) = \mathcal{E}^{t,\bar{T}}(a_{T\wedge\tau_t^\star}(C)) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{me}}(a_{\tau_t^\star}(C))$ ■

The next result provides an alternative characterization of the indifference price. It is consistent with the Snell envelope representation in a complete market. In addition, it confirms that the indifference price $a_t(C;T)$ is greater than any of the prices $\mathcal{E}^{t,s}_{\mathbb{Q}^{me}}(C_s)$ of the corresponding European claims.

Theorem 34 *Define a sequence of $\mathcal{F}_t$-measurable variables ν_t as follows:*

$$\nu_t = \sup_{t\leq\tau\leq\bar{T}} \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{me}}(C_\tau). \tag{5.91}$$

The indifference price process $a_t(C;T)$ and the process ν_t coincide.

Proof. For any stopping time τ, $a_\tau(C;T) \geq C_\tau$. The operators $\mathcal{E}^{t,t+1}_{\mathbb{Q}^{me}}(\cdot)$ are monotone and Proposition 5.2 yields $a_t(C;T) \geq \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{me}}(a_\tau(C)) \geq \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{me}}(C_\tau)$. On the other hand, $a_{\tau_t^\star}(C) = C_{\tau_t^\star}$. Therefore, $a_t(C;T) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{me}}(a_{\tau_t^\star}(C)) = \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{me}}(C_{\tau^\star}) = \sup\limits_{t\leq\tau\leq T} \mathcal{E}^{t,\bar{T}}_{\mathbb{Q}^{me}}(C_\tau)$ ■

Theorem 35 (Optimal stopping time) *Optimal stopping time defined in equation (5.87) and $\alpha_{t+1}, ...\alpha_{\bar{T}}$ shown in equation (5.94) are a solution of stochastic optimization problem (5.53), that is*

$$V^C(X_t, S_t, Y_t, s; T) = E_{\mathbb{P}}[U_t^B\left(X_{\tau_t^\star} + C_{\tau_t^\star}, \tau_t^\star\right); T)/\mathcal{F}_t], \tag{5.92}$$

with

$$X_{\tau^\star} = \sum_{u=t}^{\tau_t^\star} (S_{u+1} - S_u)\alpha^\star_{u+1}, \tag{5.93}$$

$$
\alpha_t^\star = \frac{1}{\gamma S_{t-1}(\xi_t^u - \xi_t^d)} \log\left(\frac{(\xi_t^u - 1)\mathbb{P}(A_t/\mathcal{F}_{t-1})}{(1-\xi_t^d)\mathbb{P}(A_t^c/\mathcal{F}_{t-1})}\right) + \frac{1}{\gamma S_{t-1}(\xi_t^u - \xi_t^d)} \cdot
$$

$$
\log\left(\frac{(e^{\gamma - a_{t+1}^{uu}}\mathbb{P}(A_t, B_t/\mathcal{F}_{t-1}) + e^{\gamma - a_{t+1}^{ud}}\mathbb{P}(A_t, B_t^c/\mathcal{F}_{t-1}))\mathbb{P}(A_t^c/\mathcal{F}_{t-1})}{(e^{\gamma - a_{t+1}^{du}}\mathbb{P}(A_t^c, B_t/\mathcal{F}_{t-1}) + e^{\gamma - a_{t+1}^{dd}}\mathbb{P}(A_t^c, B_t^c/\mathcal{F}_{t-1}))\mathbb{P}(A_t/\mathcal{F}_{t-1})}\right), \tag{5.94}
$$

$$
\begin{aligned}
&\mathbb{P}(A_{t+1}/\mathcal{F}_t) = \mathbb{P}\left(S_{t+1} = S_{t+1}^u/\mathcal{F}_t\right), \\
&\mathbb{P}(A_{t+1}^c/\mathcal{F}_t) = \mathbb{P}\left(S_{t+1} = S_{t+1}^d/\mathcal{F}_t\right), \\
&\mathbb{P}(A_{t+1}B_{t+1}/\mathcal{F}_t) = \mathbb{P}\left(S_{t+1} = S_{t+1}^u, Y_{t+1} = Y_{t+1}^u/\mathcal{F}_t\right), \\
&\mathbb{P}(A_{t+1}^c B_{t+1}/\mathcal{F}_t) = \mathbb{P}\left(S_{t+1} = S_{t+1}^d, Y_{t+1} = Y_{t+1}^u/\mathcal{F}_t\right), \\
&\mathbb{P}(A_{t+1}B_{t+1}^c/\mathcal{F}_t) = \mathbb{P}\left(S_{t+1} = S_{t+1}^u, Y_{t+1} = Y_{t+1}^d/\mathcal{F}_t\right), \\
&\mathbb{P}(A_{t+1}^c B_{t+1}^c/\mathcal{F}_t) = \mathbb{P}\left(S_{t+1} = S_{t+1}^d, Y_{t+1} = Y_{t+1}^d/\mathcal{F}_t\right),
\end{aligned} \tag{5.95}
$$

and a_{t+1}^{uu}, a_{t+1}^{ud}, a_{t+1}^{du} and a_{t+1}^{dd} denoting the four possible values of $a_{t+1}(C;T)$, as viewed from time t.

Proof. We first check the result for $t = \bar{T} - 1$.

$$
\begin{aligned}
&E_{\mathbb{P}}[U_{\bar{T}-1}^B(X_{\tau_{\bar{T}-1}^\star} + C_{\tau_{\bar{T}-1}^\star}; T)/\mathcal{F}_{\bar{T}-1}] = \\
&E_{\mathbb{P}}[\mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}} U_{\tau_{\bar{T}-1}^\star}^B(X_{\tau_{\bar{T}-1}^\star} + C_{\tau_{\bar{T}-1}^\star}; T) + \\
&(1 - \mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}}) U_{\tau_{\bar{T}-1}^\star}^B(X_{\tau_{\bar{T}-1}^\star} + C_{\tau_{\bar{T}-1}^\star}; T)/\mathcal{F}_{\bar{T}-1}] = \\
&\mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}} U_{\bar{T}-1}^B(X_{\bar{T}-1} + C_{\bar{T}-1}; T) + \\
&(1 - \mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}}) E_{\mathbb{P}}[U_{\bar{T}}^B(X_{\bar{T}} + C_{\bar{T}}; T)/\mathcal{F}_{\bar{T}-1}] = \\
&\mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}} U_{\bar{T}-1}^B(X_{\bar{T}-1} + C_{\bar{T}-1}; T) + \\
&(1 - \mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}}) E_{\mathbb{P}}[V^C(X_{\bar{T}}, S_{\bar{T}}, Y_{\bar{T}}, \bar{T}; T)/\mathcal{F}_{\bar{T}-1}].
\end{aligned} \tag{5.96}
$$

By definition of $\tau_{\bar{T}-1}^\star$,

$$
\begin{aligned}
&(1 - \mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}}) E_{\mathbb{P}}[V^C(X_{\bar{T}}, S_{\bar{T}}, Y_{\bar{T}}, \bar{T}; T)/\mathcal{F}_{\bar{T}-1}] = \\
&(1 - \mathbf{1}_{\{\tau_{\bar{T}-1}^\star = \bar{T}-1\}}) V^C(X_{\bar{T}-1}, S_{\bar{T}-1}, Y_{\bar{T}-1}, \bar{T}-1; T)
\end{aligned} \tag{5.97}
$$

for $\alpha_{\bar{T}}^\star$ defined as in (5.94). Therefore

$$
E_{\mathbb{P}}[U_{\tau_{\bar{T}-1}^\star}^B(X_{\tau_{\bar{T}-1}^\star} + C_{\tau_{\bar{T}-1}^\star}; T)/\mathcal{F}_{\bar{T}-1}] = V^C(X_{\bar{T}-1}, S_{\bar{T}-1}, Y_{\bar{T}-1}\bar{T}-1; T). \tag{5.98}
$$

Assuming that (5.92) hold for $t = u+1, ..., \bar{T}$, we need to show that (5.92) also holds for $t = u$. Expand $E_{\mathbb{P}}[U^B_{\tau^\star_u}(X_{\tau^\star_u} + C_{\tau^\star_u}; T)/\mathcal{F}_u]$ as:

$$\begin{aligned} & E_{\mathbb{P}}[U^B_{\tau^\star_u}(X_{\tau^\star_u} + C_{\tau^\star_u}; T)/\mathcal{F}_u] = E_{\mathbb{P}}[\mathbf{1}_{\{\tau^\star_u = u\}} U^B_{\tau^\star_u}(X_{\tau^\star_u} + C_{\tau^\star_u}; T) + \\ & (1 - \mathbf{1}_{\{\tau^\star_u = u\}}) U^B_{\tau^\star_u}(X_{\tau^\star_u} + C_{\tau^\star_u}; T)/\mathcal{F}_u] = \mathbf{1}_{\{\tau^\star_u = u\}} U^B_u(X_u + C_u; T) + \\ & (1 - \mathbf{1}_{\{\tau^\star_u = u\}}) E_{\mathbb{P}}[U^B_{\tau^\star_u}(X_{\tau^\star_u} + C_{\tau^\star_u}; T)/\mathcal{F}_u]. \end{aligned} \tag{5.99}$$

For $\{w : \tau^\star_u(w) > u\}$, $\tau^\star_u = \tau^\star_{u+1}$ and

$$\begin{aligned} & E_{\mathbb{P}}[U^B_{\tau^\star_u}(X_{\tau^\star_u} + C_{\tau^\star_u}; T)/\mathcal{F}_u] = \mathbf{1}_{\{\tau^\star_u = u\}} V^C(X_u, S_u, Y_u, u; T) + \\ & (1 - \mathbf{1}_{\{\tau^\star_u = u\}}) E_{\mathbb{P}}[U^B_{\tau^\star_{u+1}}(X_{\tau^\star_{u+1}} + C_{\tau^\star_{u+1}}; T)/\mathcal{F}_u] = \\ & \mathbf{1}_{\{\tau^\star_u = u\}} V^C(X_u, S_u, Y_u, u; T) + (1 - \mathbf{1}_{\{\tau^\star_u = u\}}) E_{\mathbb{P}}[V^C(X_{u+1}, S_{u+1}, Y_{u+1}, u+1; T)/\mathcal{F}_u] \end{aligned} \tag{5.100}$$

by induction assumption. By definition of $\tau^\star_u$,

$$\begin{aligned} & (1 - \mathbf{1}_{\{\tau^\star_u = u\}}) E_{\mathbb{P}}[V^C(X_{u+1}, S_{u+1}, Y_{u+1}, u+1; T)/\mathcal{F}_u] = \\ & (1 - \mathbf{1}_{\{\tau^\star_u = u\}}) V^C(X_u, S_u, Y_u, u; T) \end{aligned} \tag{5.101}$$

for $\alpha^\star_{u+1}$ defined as in (5.94).
Therefore, $E_{\mathbb{P}}[U^B_{\tau^\star_u}(X_{\tau^\star_u} + C_{\tau^\star_u}; T)/\mathcal{F}_u] = V^C(X_u, S_u, Y_u, u; T)$ ■

We have presented the two indifference pricing algorithms with different dynamic utilities. Both algorithms are recursive, have s structure similar to the complete market price of an American contract, and use the same nonlinear pricing functionals $\mathcal{E}^{t,u}_Q(\cdot)$. However, there are some differences, such as the measures used for valuation and dependence on the normalization point. As with European contracts, the forward early exercise price does not depend on the forward normalization point s, nor does it depend on the end of the investment horizon T. Not so for the backward early exercise price, which uses the minimal martingale measure $\mathbb{Q}^{me}$, which depends on the backward normalization point T (also the end of the investment horizon). The next and final section of this chapter works under the reduced model assumption of corollary 10 in chapter 3, and shows how the two early exercise prices are related.

[18] recently suggested a valuation method similar to ours. This method uses the traditional static preferences of the form $-e^{-\gamma x}$ to price an early exercise liability generating cashflows on $[0; T]$. In [18] the agents preferences are fixed at time T, as they are for our backward utility process in our work. [18] works with a reduced

model, for which the minimal entropy measure and the martingale measure coincide. In addition to using dynamic preferences and having a more general model, our results are fully formulated even in the multi-period setting in terms of the corresponding theorems, and have been rigorously confirmed. We address properties of the indifference prices in a rigorous way and provide an alternative characterization of the early exercise price, as the supremum of the European prices.

5.3 Reduced model results

The concept of a reduced model was introduced in chapter 3, corollary 10. In such a model the historical distribution of the next period's traded asset value does not depend of the path of the non-traded stochastic risk factor Y, and neither do the variables ξ_{t+1}^{u} and ξ_{t+1}^{d}, for all $0 \leq t \leq T-1$. We have shown in corollary 10 of chapter 3 that in a reduced model, the minimal martingale measure and the minimal entropy measure are the same. In addition, theorem 23 of section 4.3 states that the backward and the forward European prices are the same. Naturally, the American backward and forward prices are the same as well. The results are stated without a proof since the latter is obvious once the American pricing algorithms 31 and 25 have been established.

Theorem 36 *In the reduced binomial model, i.e. when*

$$\mathbb{P}(\xi_{t+1}/\mathcal{F}_t) = \mathbb{P}(\xi_{t+1}/\mathcal{F}_t^S), \; t = 0, 1, ..., T-1, \tag{5.102}$$

and ξ_{t+1}^{u}, ξ_{t+1}^{d} *are* $\mathcal{F}_t^S$*-measurable, the forward and backward indifference prices of the American contract* C *written at* t_0 *and maturing at* $\bar{T}$*, coincide:*

$$a_t^F(C) = a_t^B(C; T) \tag{5.103}$$

for $t_0 \leq t \leq \bar{T} \leq T$.

The next result assumes the reduced model and considers the payoff depending on the traded asset only. In that case both the backward and the forward prices coincide, and the call option on the traded asset S should never be exercised. In the non-reduced model, ξ_{t+1}^{u} and ξ_{t+1}^{d} may be dependent on the non-traded risk factor values. Then, under any equivalent to $\mathbb{P}$ martingale measure Q, the distribution of the next period's traded asset value S_{t+1} would depend on Y. In that case, even if the payoff only depends on the traded asset S, the pricing algorithm cannot be simplified to pricing using expectations. Then, the call option on the traded asset may have to be exercised early. The proof of the next result follows easily from proposition 4.5 of section 4.3, theorems 31 and 25, and Jensen's inequality for convex functions, and therefore is omitted.

Proposition 5.3 *Under the reduced model assumption of corollary 10, chapter 3 (also shown in theorem 36 above), if the intrinsic payoff $C(S_t, Y_t)$ does not depend on Y_t for all $t_0 \leq t \leq \bar{T}$, then*

$$a_t^F(C) = a_t^B(C; T) \tag{5.104}$$

In addition, if $C(S_t) = (S_t - K)^+$, then

$$a_t^F(C) = a_t^B(C; T) = E_{\mathbb{Q}^{mm}}[(S_{\bar{T}} - K)^+ / \mathcal{F}_t^S] = E_{\mathbb{Q}^{me}}[(S_{\bar{T}} - K)^+ / \mathcal{F}_t^S], \tag{5.105}$$

for any $t_0 \leq t \leq \bar{T}$.

Chapter 6

The continuous time model and the dynamic indifference valuation.

In this chapter we focus our study on valuation of early exercise contracts in continuous time. The binomial model and results presented above provide an explicit algorithm for valuation of early exercise and partial exercise claims with either backward of forward utility, including the cases when dynamics of the traded asset are affected by the non-traded factor. Unfortunately for certain models, the associated binomial tree would not recombine, making the practical implementation computationally intractable, as discussed in more detail in the concluding chapter. For those models, the more straight forward way to value early exercise contracts with non-traded assets would be in the PDE framework, developed herein.

In modeling the traded asset and the non-traded stochastic factor, a typical setting would be the one with the two SDEs governing the dynamics of the instruments, respectively. The dependence of the stock dynamics on the non-traded factor can show itself in two distinct ways: through the correlation coefficient between the respective Brownian motions and through explicit functional dependence of stock's SDE coefficients. If only the first type is present, then the Sharpe ratio of the traded asset does not depend on the value of a stochastic factor. Such a model was called an *almost complete* model by [45]. When both types of dependence are present, the valuation becomes more involved, as we see further.

Several studies have already been initiated to value early exercise claims with non-traded assets in continuous time. For us, the closest related are the ones that apply indifference pricing methodology. Among those are a number of studies that used the traditional, static exponential utility $U(x) = -e^{-\gamma x}$ and the constant Sharpe ratio. Among the ones are [20], [14], [22], [44], [34].

In [20], perpetual options are considered. In [22] and [34], the partial exercise feature is brought in. In our study we assume that that early exercise option is expiring at a certain time and, at that time, it has to be exercised if not exercised so far. Partial exercise has been addressed in our work in discrete time. In continuous time, we do not address the partial exercise feature. Intead, we focus on working with more general model dynamics, and are going to derive the value for the American contract with finite-horizon under two different dynamic utilities. Extending our continuous time results even more, to partial exercise or to infinite horizon, is a future research topic.

With more general models, valuation of early exercise claims with non-tradable assets in continuous time is represented by the works of, for example, [32] and [31]. There, the authors suggest to work with price processes given as semi-martingales and deploy game-theoretic arguments to deduce the price, but those results again either use a static exponential utility or are not as explicit as the ones we present below.

In what follows, our goal is to derive the price of the early exercise claim with non-traded stochastic factor, for the forward and the backward preferences, as solutions of their respective variational inequalities. We start by presenting the corresponding continuous time results for European contracts, first for the forward utility, then for the backward. [42] work with writer's value. We are pricing American claims from the buyer's prospective. All the concepts and results derived below will be stated from the buyer's point of view. We use classical stochastic control arguments together with existing results about the Backward and the Forward utility processes and the associated measures. We comment on the similarities and the differences in the representations for the Backward and Forward prices. We present the alternative characterizations for the Backward and Forward prices, using the nonlinear functionals under the minimal entropy and minimal martingale measures correspondingly. We comment on the similarities between the alternative representations and results obtained in our binomial model.

6.1 Market model

We use a market model consisting of a riskless bond, a risky traded stock and a risky (non-traded) stochastic factor. The interest rate is assumed to be zero. Therefore, the bond price B_s is identically 1 at all times. The stock price process, S_s is a diffusion process. Its drift and volatility are dependent on the level of the non-traded factor Y_s. The non-traded risk factor itself, Y_s, is also a diffusion. S_s and Y_s satisfy the corresponding pair of stochastic differential equations:

$$\begin{cases} dS_s = \mu(Y_s, s)S_s ds + \sigma(Y_s, s)S_s dW_s^1, \\ dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s^2. \end{cases} \tag{6.1}$$

The processes W_s^1 and W_s^2 are standard Brownian motions defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_s, \mathbb{P})$. $\mathcal{F}_s$ is the augmented σ-algebra generated by $(W_u^1, W_u^2, 0 \leq u \leq s)$. The Brownian motions are correlated with a correlation coefficient $\rho \in (-1, 1)$. μ,σ, a and b are such that (6.1) has unique strong solution. A claim C is written on a both assets, and could not be perfectly replicated through trading. As for now, the claim is assumed to be European.

An investor is endowed with the time t wealth of x dollars, and uses a self-financing trading strategy π_s, $t \leq s \leq T$. Assuming zero interest rate, it is not difficult to see that the wealth of the investor evolves according to the SDE shown below:

$$\begin{cases} dX_s = \pi_s \mu(Y_s, s)ds + \pi_s \sigma(Y_s, s)dW_s^1, \\ X_t = x. \end{cases} \tag{6.2}$$

The trading strategy π_s is assumed be such that $E\int_0^T \sigma_s^2 \pi_s^2 ds < \infty$. The set of all such trading strategies forms the set $\mathcal{A}$ of admissible controls. The sopping time τ, taking values in $[t_0; \bar{T}]$, represents the exercise time chosen by the investor. $\bar{T}$ is the expiration date of the claim.

6.2 Forward dynamic preferences and indifference price for European contracts

In this section we state results of [42] obtained for the forward utility process for European claims. Those results are necessary for comparison with our discrete time formulas obtained in section 4.1, and set the modelling framework and create a foundation for valuation of American contracts with forward dynamic utility, that are presented in subsequent sections.

The agent's preferences are specified by the forward dynamic utility of the form:

$$U_t^F(x;s) = -e^{-\gamma x + h(s,t)}, \tag{6.3}$$

with

$$h(s,t) = \int_t^s \frac{1}{2}\lambda^2(Y_u)du \tag{6.4}$$

and λ being the Sharpe ratio:

$$\lambda_u = \lambda(Y_u) = \frac{\mu(Y_u)}{\sigma(Y_u)}. \tag{6.5}$$

The time point s is referred to as the normalization point, that is at $t = s$,

$$U_s^F(x;s) = -e^{-\gamma x}. \tag{6.6}$$

An important property of the forward dynamic utility that it is self-generating, that is

$$U_t^F(x;s) = \sup_{\mathcal{A}} E_{\mathbb{P}}[U_T^F(X_T;s)/\mathcal{F}_t], t \geq s. \tag{6.7}$$

The properties of self-generation (6.7) and the normalization (6.6) are satisfied by several different processes, one of them being the forward utility process (6.3). Another process, as [42] point out, would be, for example of the form:

$$\begin{cases} U_t^F(x;s) = -e^{-\gamma x - Z(s,t)}, \\ Z(s,t) = \int_s^t \frac{1}{2}\lambda^2(Y_u)du + \int_s^t \lambda(Y_u)dW_u^1. \end{cases} \tag{6.8}$$

The characterization of the class of processes that satisfy the the self-generation condition and the normalization condition together, remains an open question, as well as what are the conditions under which (6.7) and (6.6) yield unique solution. We are not addressing this question herein.

The self generation properly implies that the value function (defined traditionally, as the expectation of the agent's terminal wealth) coincides with the forward dynamic utility process itself. Another attractive property of the forward utility process (6.3), is that is is independent of the end-point T of investment horizon. As will be seen later, the prices generated with the forward dynamic utility process (6.3) are independent of T as well. Below we continue with a summary of the relevant results obtained by [42] regarding the indifference prices generated by forward dynamic utility and the relevant measures, the minimal entropy and the minimal martingale measure, defined correspondingly the minimizers of the two entropic functionals:

Definition 6.1 (Minimal Entropy measure)

$$\mathcal{H}\left(\mathbb{Q}^{me}/\mathbb{P}\right) = \min_{Q \in \mathcal{Q}_e} E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}}\right]. \tag{6.9}$$

Definition 6.2 (Minimal Martingale measure)

$$\mathcal{H}\left(\mathbb{Q}^{mm}/\mathbb{P}\right) = \min_{Q \in \mathcal{Q}_e} E_{\mathbb{P}}\left[- \log \frac{dQ}{d\mathbb{P}}\right]. \tag{6.10}$$

The Minimal Entropy measure and the Minimal Martingale measure defined above possess the following Radon-Nikodym densities:

$$\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}} = \exp\left(-\int_0^T \lambda_u dW_u^1 - \int_0^T \frac{1}{2}\lambda_u^2 du\right) \tag{6.11}$$

and

$$\frac{d\mathbb{Q}^{me}}{d\mathbb{P}} = \exp\left(-\int_0^T \lambda_u dW_u^1 - \int_0^T \hat{\lambda}_u dW_u^{1,\perp} - \int_0^T \frac{1}{2}(\lambda_u^2 + \hat{\lambda}_u^2) du\right), \tag{6.12}$$

with $dW_u^2 = \rho dW_u^1 + \sqrt{1-\rho^2} dW_u^{1,\perp}$. dW_u^1 and $dW_u^{1,\perp}$ are the two orthogonal Brownian motions.

The minimal aggregate relative conditional (on time t) entropy $\mathcal{H}(\mathbb{Q}^{me}/\mathbb{P})$ equals

$$\mathcal{H}(\mathbb{Q}^{me}/\mathbb{P}) = -\mathcal{J}_{\mathbb{Q}^{mm}}\left(-\int_t^T \frac{1}{2}\lambda_u^2 du\right) \tag{6.13}$$

with $\mathcal{J}_Q(\cdot)$ being the conditional nonlinear expectation of a generic random variable

$Z \in \mathcal{F}_T$, defined with respect to any equivalent martingale measure Q as

$$\mathcal{J}_Q(Z) = \frac{1}{1-\rho^2} \ln E_Q[e^{(1-\rho^2)Z}/\mathcal{F}_t]. \tag{6.14}$$

We point out that $\mathcal{J}_Q(\cdot)$ is independent of γ. We also define another nonlinear functional, $\mathcal{E}_Q(\cdot)$ as

$$\mathcal{E}_Q(Z) = -\frac{1}{\gamma(1-\rho^2)} \ln E_Q[e^{-\gamma(1-\rho^2)Z}/\mathcal{F}_t]. \tag{6.15}$$

Clearly, $\mathcal{J}_Q(Z) = -\gamma\mathcal{E}_Q(-\frac{1}{\gamma}Z)$, or $\mathcal{J}_Q(Z)$ is the same as $\mathcal{E}_Q(Z)$ with $\gamma = -1$. The functional $\mathcal{E}(\cdot)$ does depend on the level of the absolute risk aversion.

The relative aggregate conditional (on time t) entropy of the minimal martingale measure $\mathcal{H}(\mathbb{Q}^{mm}/\mathbb{P})$ equals:

$$\mathcal{H}(\mathbb{Q}^{mm}/\mathbb{P}) = E_{\mathbb{Q}^{mm}}\left[\int_t^T \frac{1}{2}\lambda_u^2 ds\right]. \tag{6.16}$$

As shown in [42], the minimal aggregate relative entropy process is given by

$$\mathcal{H}(\mathbb{Q}^{me}/\mathbb{P}) = E_{\mathbb{Q}^{me}}\left[\int_t^T \frac{1}{2}\left(\lambda^2(Y_s) + \hat{\lambda}^2(Y_s, s; T)\right) ds/\mathcal{F}_t\right], \tag{6.17}$$

with λ being the Sharpe ratio as in (6.5), $\hat{\lambda}$: $\mathcal{R} \times [0; T] \to \mathcal{R}^+$ defined as

$$\hat{\lambda}(y, s; T) = -\frac{1}{\sqrt{1-\rho^2}} a(y) \frac{f_y(y, t; T)}{f(y, t; T)}, \tag{6.18}$$

and with f solving

$$\begin{cases} f_t + \frac{1}{2}a^2(y)\frac{\partial^2 f}{\partial y^2} + (b(y) - \rho\lambda(y)a(y))\frac{\partial f}{\partial y} = \frac{1}{2}(1-\rho^2)\lambda^2(y) f, \\ f(y, T) = 1. \end{cases} \tag{6.19}$$

So far we have provided two different representations for the minimal aggregate entropy process $\mathcal{H}(\mathbb{Q}^{me}/\mathbb{P})$, that is equations (6.13) and (6.17). Both of those representations could be written equivalently in the PDE form, as the next proposition shows.

Proposition 6.1 *The minimal aggregate entropy process equals*

$$\mathcal{H}(\mathbb{Q}^{me}/\mathbb{P}) = \tilde{H}(Y_t, t; T) \tag{6.20}$$

with the function $\tilde{H}(y, t; T) : \mathcal{R} \times [0; T] \to \mathcal{R}$ *solves either of the two PDEs below:*

$$\begin{cases} \tilde{H}_t + \mathcal{L}^{mm}\tilde{H} - \frac{1}{2}(1-\rho^2)a^2(y)\tilde{H}_y^2 + \frac{1}{2}\lambda^2(y) = 0, \\ \tilde{H}(y, T; T) = 0. \end{cases} \tag{6.21}$$

or

$$\begin{cases} \tilde{H}_t + \mathcal{L}^{me}\tilde{H} + \frac{1}{2}\left(\lambda^2(y) + \hat{\lambda}^2(y)\right) = 0, \\ \tilde{H}(y, T; T) = 0. \end{cases} \tag{6.22}$$

The operators $\mathcal{L}^{mm}$ *and* $\mathcal{L}^{me}$ *are defined below in equations (6.23) and (6.24) correspondingly:*

$$\begin{aligned} \mathcal{L}^{mm} &= \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2} + \rho\sigma(y)Sa(y)\frac{\partial^2}{\partial S\partial y} + \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2} \\ &+ (b(y) - \rho\lambda(y)a(y))\frac{\partial}{\partial y}, \end{aligned} \tag{6.23}$$

$$\begin{aligned} \mathcal{L}^{Y,me} &= \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2} + \rho\sigma(y)Sa(y)\frac{\partial^2}{\partial S\partial y} + \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2} \\ &+ \left(b(y) - \rho\lambda(y)a(y) + a^2(y)\frac{f_y(y,t;T)}{f(y,t;T)}\right)\frac{\partial}{\partial y} \end{aligned} \tag{6.24}$$

and f *solves equation (6.19).*

We are now ready to define the indifference price and state its corresponding PDE.

Definition 6.3 (Forward European indifference price) *Let* $s \geq 0$ *be the forward normalization point and consider a claim* $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$*, written at* $t_0 \geq s$ *and maturing at* $\bar{T} < T$*. For* $t \in [t_0; \bar{T}]$*, the forward indifference value process* $\nu_t^F(C_{\bar{T}}; s)$ *satisfies the pricing equation:*

$$U_t^F(x + \nu_t^F(C_{\bar{T}}; s); s) = \sup_{\mathcal{A}} E\left[U_{\bar{T}}^F(X_{\bar{T}} + C_{\bar{T}}; s)/\mathcal{F}_t\right], \tag{6.25}$$

for all $x \in \mathcal{R}$*,* $X_t = x$*.*

Proposition 6.2 *The forward indifference price* $\nu_t^F(C_{\bar{T}}; s)$ *equals:*

$$\nu_t^F(C_{\bar{T}}; s) = p^F(S_t, Y_t, t), \tag{6.26}$$

with function $p^F(S,y,t) : \mathcal{R}^+ \times \mathcal{R} \times [t_0; \bar{T}] \to \mathcal{R}$ solving the PDE:

$$\begin{cases} p_t^F + \mathcal{L}^{mm} p^F - \frac{1}{2}\gamma(1-\rho^2)a(y)^2(p_y^F)^2 = 0, \\ p^F(S,y,\bar{T}) = C(S,y), \end{cases} \tag{6.27}$$

and the operator $\mathcal{L}^{mm}$ defined as in (6.23) above.

6.3 Backward utility process and indifference price for European contracts

This section introduces the backward utility process, following closely the work of [42], and re-stating their results to reflect the buyer's point of view. Those results lay the foundation for valuing American claim that we attempt in the next two sections.

As the with the forward utility, the backward utility process is created through the introduction of the two necessary condition, the self-generation and the normalization. Unlike the forward utility, that is normalized at a time point in the past, the backward utility is normalized at the future time point T, so that

$$U_T^B(x;T) = -e^{-\gamma x}. \tag{6.28}$$

The self-generation condition for the backward utility is as follows:

$$U_t^B(x;T) = \sup_{\mathcal{A}} E_{\mathbb{P}}[U_T^B(X_T;T)/\mathcal{F}_t], t \geq s. \tag{6.29}$$

The normalization (6.28) and the self-generation (6.29) imply the unique backward dynamic utility process of the form:

$$U_t^B(x;T) = -e^{-\gamma x - \tilde{H}(y,t;T)}, t \leq T \tag{6.30}$$

As one can see, the backward utility process is actually the same as what is traditionally called "the plain investment value function" of an agent maximizing his expected utility of the terminal wealth, when the agent is endowed with static exponential utility. We next state the definition of the backward indifference price and the corresponding PDE it solves.

Definition 6.4 (Backward European indifference price) *Let $\bar{T} < T$ be the maturity of the European contract $C_{\bar{T}}$, written at time t_0. For $t \in [t_0; \bar{T}]$, the backward indifference price process $\nu_t^B(C_{\bar{T}}; T))$ satisfies the pricing condition:*

$$U_t^B(x + \nu_t^B(C_{\bar{T}}; T)) = \sup_{\mathcal{A}} E\left[U_{\bar{T}}^B(X_{\bar{T}} + C_{\bar{T}}; T)/\mathcal{F}_t\right], \tag{6.31}$$

for all $x \in \mathcal{R}$, $X_t = x$.

Proposition 6.3 *The backward indifference price process $\nu_t^B(C_{\bar{T}}; T)$ is given by:*

$$\nu_t^B(C_{\bar{T}}; T) = p^B(S_t, Y_t, t), \tag{6.32}$$

where function $p^B : \mathcal{R}^+ \times \mathcal{R} \times [t_0; \bar{T}] \to \mathcal{R}$ satisfies:

$$\begin{cases} p_t^B + \mathcal{L}^{me} p^B - \frac{1}{2}\gamma(1-\rho^2)a(y)^2(p_y^B)^2 = 0, \\ p^B(S, y, \bar{T}) = C(S, y). \end{cases} \tag{6.33}$$

The operator $\mathcal{L}^{me}$ is the one defined in (6.24) above.

Comparing results of propositions 6.3 and 6.2, one could easily see that the two pricing PDE equations have the same terminal conditions, the same structure, but different operators, $\mathcal{L}^{mm}$ and $\mathcal{L}^{me}$. The two operators differ only in the drift term, the latter having the same drift as $\mathcal{L}^{mm}$, but with an extra added factor $a^2(y)\dfrac{f_y(y,t;T)}{f(y,t;T)}$. In our model, when the Sharpe ratio $\lambda(y)$ is independent of y, the Sharpe ratio is constant, i.e., the model becomes *a*lmost complete when the Sharpe ratio is constant. The PDE for function f has a unique solution. One could see that, having a constant λ, implies that f is identically 1, and that the operators $\mathcal{L}^{mm}$ and $\mathcal{L}^{me}$ then become the same, as become the forward and the backward prices. The difference between the backward and the forward price could only be seen when λ has some dependence on y.

We provide another representation of the backward and forward prices, in the form of the nonlinear functionals.

Proposition 6.4 *The Backward and the Forward buyer's indifference prices of the*

European contract $C_{\bar{T}}$ *equal*

$$\begin{aligned} \nu_t^B(C_{\bar{T}}; T) &= \mathcal{E}_{\mathbb{Q}^{me}}\left(C_{\bar{T}}\right), \\ \nu_t^F(C_{\bar{T}}; s) &= \mathcal{E}_{\mathbb{Q}^{mm}}\left(C_{\bar{T}}\right), \end{aligned} \tag{6.34}$$

with the nonlinear functional $\mathcal{E}_Q(\cdot)$ *defined in equation (6.15) of the previous section for any martingale measure* Q.

6.4 Forward indifference price for American contracts

Traditionally, in continuous time the indifference price is represented with PDEs. In this section we first extend the work of [42] by deriving the forward indifference value of an American contracts as the solution of the corresponding variational inequality. Motivated by our discrete time results in earlier chapters(see 28 in section 5.1), we provide an alternative representation using nonlinear functionals $\mathcal{E}_Q(\cdot)$, in the spirit of proposition 6.4 of the previous section. This alternative representation provides a convenient way for comparison to the discrete time pricing fomulas.

The buyer is now allowed to exercise his contract at any time within the interval $[t_0; \bar{T}]$, and a stopping time τ represents the exercise time of the contract, chosen by the buyer. Upon exercise, the buyer receives the payoff $X_\tau + C_\tau$. We formulate the indifference pricing condition similarly to the way [42] formulated it for the European contracts, but with the stopping time τ taken into account.

Definition 6.5 (Forward early exercise indifference price) *Let* $\bar{T}$ *be the expiration date of the American contract with intrinsic payoff* C_t*, written at time* t_0*,* $t_0 \le t \le \bar{T}$. *For* $t \in [t_0; \bar{T}]$*, the forward early exercise indifference price process* $\nu_t^{a,F}(C; s)$ *satisfies the pricing condition:*

$$U_t^F(x + \nu_t^{a,F}(C; s); s) = \sup_{\mathcal{A},\tau} E_{\mathbb{P}}\left[U_\tau^F(X_\tau + C_\tau; s)/\mathcal{F}_t\right]. \tag{6.35}$$

for all $x \in \mathcal{R}$*,* $X_t = x$.

Below we present the main result of this section.

Theorem 37 *The forward early exercise indifference price* $\nu_t^{a,F}(C; s)$ *satisfies:*

$$\nu_t^{a,F}(C; s) = \nu^{a,F}(S, y, t), \tag{6.36}$$

with function $\nu^{a,F}(S,y,t) : \mathcal{R}^+ \times \mathcal{R} \times [t_0;\bar{T}] \to \mathcal{R}$ being the unique bounded viscosity solution of the quasilinear variational inequality shown below:

$$\begin{cases} \min\{-\nu_t^{a,F} - \mathcal{L}^{mm}\nu^{a,F} + \frac{1}{2}\gamma(1-\rho^2)a(y)^2(\nu_y^{a,F})^2, \nu^{a,F} - C(S,y)\} = 0, \\ \nu^{a,F}(S,y,\bar{T}) = C(S,y). \end{cases} \tag{6.37}$$

Proof. We assume that indifference prices and other functions we define and use in this proof are sufficiently differentiable. The formal arguments presented herein could be made rigorous in the viscosity sense. First note that, using measurability property of $h(s,t)$,

$$\sup_{\mathcal{A},\tau} E_{\mathbb{P}}\left[U_\tau^F(X_\tau + C_\tau; s)/\mathcal{F}_t\right] = e^{\int_s^t \frac{1}{2}\lambda^2(Y_u)du} \sup_{\mathcal{A},\tau} E_{\mathbb{P}}[-e^{-\gamma(X_\tau + C_\tau) + \int_t^\tau \frac{1}{2}\lambda^2(Y_u)du}/\mathcal{F}_t] = \\ e^{\int_s^t \frac{1}{2}\lambda^2(Y_u)du} u^c(X_t, S_t, Y_t, t). \tag{6.38}$$

Using the classical stochastic stochastic control arguments for problems with optimal stopping, one could see that function $u^c(x,S,y,t) : \mathcal{R} \times \mathcal{R}^+ \times \mathcal{R} \times [t_0;\bar{T}] \to \mathcal{R}$ satisfies the following equation:

$$\begin{cases} \min\{-u_t^c - \max_\pi \left(\frac{1}{2}\sigma^2(y)\pi^2 u_{xx}^c + \pi(\sigma^2(y)Su_{xS}^c + \rho\sigma(y)a(y)u_{xy}^c + \mu(y)u_x^c\right) \\ -\mathcal{L}^{(S,y)}u^c - \frac{1}{2}\lambda^2(y)u, u^c + e^{-\gamma(x+C(S,y))}\} = 0, \\ u^c(x,S,y,\bar{T}) = -e^{-\gamma(x+C(S,y))}, \end{cases} \tag{6.39}$$

with the operator $L^{(S,y)}$ defined as:

$$\begin{aligned} \mathcal{L}^{(S,y)} = &\frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2} + \rho a(y)\sigma(y)S\frac{\partial^2}{\partial S\partial y} + \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2} \\ &+\mu(y)S\frac{\partial}{\partial S} + b(y)\frac{\partial}{\partial y}. \end{aligned} \tag{6.40}$$

For the forward utility process, using the self-generation condition,

$$\begin{aligned} U_t^F(x;s) &= \sup_{\mathcal{A}} E_{\mathbb{P}}[U_{\bar{T}}^F(X_{\bar{T}};s)/\mathcal{F}_t] = \\ &= e^{\int_s^t \frac{1}{2}\lambda^2(Y_u)du} \sup_{\mathcal{A}} E_{\mathbb{P}}\left[-e^{-\gamma X_{\bar{T}} + \int_t^{\bar{T}} \frac{1}{2}\lambda^2(Y_u)du}/\mathcal{F}_t\right] = e^{\int_s^t \frac{1}{2}\lambda^2(Y_u)du} u^0(X_s, Y_s, t). \end{aligned} \tag{6.41}$$

Again, using classical stochastic control arguments, $u^0(x,y,t):\mathcal{R}\times\mathcal{R}\times[t_0;\bar{T}]\to\mathcal{R}$ satisfies the following HJB equation:

$$\begin{cases} u_t^0+\max\limits_{\pi}(\frac{1}{2}\sigma^2(y)\pi^2u_{xx}^0+\pi(\sigma^2(y)Su_{xS}^0+\rho\sigma(y)a(y)u_{xy}^0+\mu(y)u_x^0) \\ +\mathcal{L}^{(y)}u^0+\frac{1}{2}\lambda^2(y)u^0=0, \\ u^0(x,y,\bar{T})=-e^{-\gamma x}, \end{cases} \tag{6.42}$$

with $\mathcal{L}^{(y)}$ as in:

$$\mathcal{L}^{(y)}=\frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2}+b(y)\frac{\partial}{\partial y}. \tag{6.43}$$

In order for the pricing condition to hold, we must have

$$u^c(X_t,S_t,Y_t,t)=u^0(X_t+\nu^{a,F}(S_t,Y_t,t),Y_t,t). \tag{6.44}$$

Let $\nu^{a,F}:\mathcal{R}^+\times\mathcal{R}\times[t_0;\bar{T}]\to\mathcal{R}$ be an arbitrary (sufficiently differentiable) function of S, y and t. Condition (6.44) then translates into:

$$u^c(x,S,y,t)=u^0(x+\nu^{a,F}(S,y,t),y,t). \tag{6.45}$$

Following the arguments in [40], plugging the right-hand side of (6.45) into equation (6.39), and taking (6.42) into account, we deduce that $\nu^{a,F}$ solves:

$$\begin{aligned} &\min\{u_x^0(x+\nu^{a,F}(S,y,t),y,t)\left(-\nu_t^{a,F}-\mathcal{L}^{mm}\nu^{a,F}\right) \\ &-u_{xy}^0(x+\nu^{a,F}(S,y,t),y,t)a^2(y)(1-\rho^2)\nu_y^{a,F} \\ &-u_{xx}^0(x+\nu^{a,F}(S,y,t),y,t)\tfrac{1}{2}a^2(y)(1-\rho^2)(\nu_y^{a,F})^2, \\ &u^0(x+\nu^{a,F}(S,y,t),y,t)+e^{-\gamma(x+C(S,y))}\}=0. \end{aligned} \tag{6.46}$$

[42] show that equation (6.42) has unique solution, which equals $u^0(x,S,y,t)=-e^{-\gamma x}$. Therefore,

$$\begin{aligned} &u_x^0(x+\nu^{a,F}(S,y,t),y,t)=\gamma e^{-\gamma(x+\nu^{a,F}(S,y,t))}, \\ &u_{xx}^0(x+\nu^{a,F}(S,y,t),y,t)=-\gamma^2e^{-\gamma(x+\nu^{a,F}(S,y,t))}, \\ &u_{xy}^0(x+\nu^{a,F}(S,y,t),y,t)=0, \\ &u^0(x+\nu^{a,F}(S,y,t),y,t)=-e^{-\gamma(x+\nu^{a,F}(S,y,t))}. \end{aligned} \tag{6.47}$$

Using equation (6.47) in (6.46), we get:

$$\begin{aligned}&\min\{\gamma e^{-\gamma(x+\nu^{a,F}(S,y,t))}\left(-\nu_t^{a,F}-\mathcal{L}^{mm}\nu^{a,F}+\tfrac{1}{2}\gamma a^2(y)(1-\rho^2)(\nu_y^{a,F})^2\right),\\&e^{-\gamma(x+\nu^{a,F}(S,y,t))}\left(-1+e^{-\gamma(C(S,y)-\nu^{a,F}(S,y,t))}\right)\}=0.\end{aligned} \tag{6.48}$$

Equation 6.48 above is equivalent to (6.37) ■

The next theorem provides another representation of the indifference price, in the form of non-linear functionals $\mathcal{E}_Q(\cdot)$ defined in (6.15). The proof follows closely Proposition 11 in [40] and is omitted.

$$\nu_t^{a,F}(C;s)=\sup_{t_0\le\tau\le\bar{T}}\left(\mathcal{E}_{\mathbb{Q}^{mm}}(C(S_\tau,Y_\tau))\right). \tag{6.49}$$

The result shown above exists in a similar form in our binomial setting, see section 5.1 theorem 28. In both cases, the continuous and the binomial, the functionals characterizing the indifference value are nonlinear, become linear as risk aversion approaches 0, use the same (minimal) martingale measure, and incorporate the stopping time in the same way, by taking the supremum of all possible buyer's prices obtained for different exercise times. A small difference is that in continuous time the instantaneous correlation coefficient is incorporated into the formula explicitly, but in the discrete time it enter the formula implicitly through the joint distribution of the traded asset and the non-traded factor.

6.5 Backward Indifference price for American contracts

In this section we derive the formulas for the backward indifference price. We start in the same way as with previous section, by introducing the pricing condition in the traditional way, but with the new backward utility process.

Definition 6.6 (Backward early exercise indifference price) *Let $\bar{T}<T$ be the expiration date of the American contract with intrinsic payoff C_t, written at time t_0, $t_0\le t\le\bar{T}$. For $t\in[t_0;\bar{T}]$, the backward early exercise indifference price process $\nu_t^{a,B}(C;T)$ satisfies the pricing condition:*

$$U_t^B(x+\nu_t^{a,B}(C;T);T)=\sup_{\mathcal{A},\tau}E_{\mathbb{P}}[U_\tau^B(X_\tau+C_\tau;T)/\mathcal{F}_t], \tag{6.50}$$

for all $x \in \mathcal{R}$, $X_t = x$.

Theorem 38 *The backward early exercise indifference price $\nu_t^{a,B}(C;T)$ satisfies:*

$$\nu_t^{a,B}(C;T) = \nu^{a,B}(S_t, Y_t, t), \tag{6.51}$$

with function $\nu^{a,B}(S,y,t) : \mathcal{R}^+ \times \mathcal{R} \times [t_0;\bar{T}] \to \mathcal{R}$ being the unique bounded viscosity solution of the following variational inequality:

$$\begin{cases} \min\{-\nu_t^{a,B} - \mathcal{L}^{me}\nu^{a,B} + \frac{1}{2}\gamma(1-\rho^2)a(y)^2(\nu_y^{a,B})^2, \nu^{a,B} - C(S,y)\} = 0, \\ \nu^{a,B}(S,y,\bar{T}) = C(S,y). \end{cases} \tag{6.52}$$

with $\mathcal{L}^{me}$ defined earlier in (6.24).

Proof. We assume that indifference prices and other functions we define and use in this are sufficiently differentiable. The formal arguments presented herein could be made rigorous in the viscosity sense. As shown in [42], the backward utility process satisfies:

$$U_t^B(X_t;T) = v^0(X_t, Y_t, t), \tag{6.53}$$

where function $v^0(x,y,t) : \mathcal{R} \times \mathcal{R} \times [0;T] \to \mathcal{R}$ equals:

$$v^0(x,y,t) = -e^{-\gamma x + \tilde{H}(y,t;T)}, \tag{6.54}$$

where the process $\tilde{H}(Y_t,t;T)$ represents the aggregate minimal entropy and the function $\tilde{H}(y,t;T)$ solves either of equations (6.21) or (6.22) defined above. On the other hand, following the classical control arguments, $v^0(x,y,t)$ satisfies the following HJB equation on $[t_0;\bar{T}]$:

$$\begin{cases} v_t^0 + \max_{\pi}(\frac{1}{2}\sigma^2(y)\pi^2 v_{xx}^0 + \pi(\sigma^2(y)Sv_{xS}^0 + \rho\sigma(y)a(y)v_{xy}^0 + \mu(y)v_x^0) \\ +\mathcal{L}^{(y)}v^0 = 0, \\ v^0(x,S,y,\bar{T}) = -e^{-\gamma x - \tilde{H}(y,\bar{T};T)} \end{cases} \tag{6.55}$$

with $\mathcal{L}^y$ defined in (6.43). Using the classical stochastic control arguments, we deduce that the right hand side of (6.50) equals:

$$\sup_{\mathcal{A},\tau} E_{\mathbb{P}}[U_\tau^B(X_\tau + C_\tau;T)/\mathcal{F}_t] = v(X_t, S_t, Y_t, t), \tag{6.56}$$

where function $v(x,S,y,t) : \mathcal{R} \times \mathcal{R}^+ \times \mathcal{R} \times [t_0;\bar{T}] \to \mathcal{R}$ is the unique bounded viscosity solution of the following variational inequality:

$$\begin{cases} \min\{-v_t - \max_{\pi}\left(\frac{1}{2}\sigma^2(y)\pi^2 v_{xx} + \pi(\sigma^2(y)Sv_{xS} + \rho\sigma(y)a(y)v_{xy} + \mu(y)v_x\right) \\ -\mathcal{L}^{(S,y)}v, v - v^0(x + C(S,y), y, t)\} = 0, \\ v(x,S,y,\bar{T}) = v^0(x + C(S,y), y, \bar{T}), \end{cases} \tag{6.57}$$

In order for the pricing condition (6.50) to hold, we have to have that

$$v(x,S,y,t) = v^0(x + \nu^{a,B}(S,y,t), y, t), \quad t \in [t_0;\bar{T}]. \tag{6.58}$$

Plugging the right hand side of equation (6.58) into the HJB equation (6.57), and taking (6.55) into account we get:

$$\begin{array}{l} min\{v^0_x(x + \nu^{a,B}(S,y,t), S, y, t)\left(-\nu^{a,B}_t - \mathcal{L}^{mm}\nu^{a,B}\right) \\ -v^0_{xy}(x + \nu^{a,B}(S,y,t), S, y, t)a^2(y)(1-\rho^2)\nu^{a,B}_y \\ -v^0_{xx}(x + \nu^{a,B}(S,y,t), S, y, t)\frac{1}{2}a^2(y)(1-\rho^2)(\nu^{a,B}_y)^2, \\ v^0(x + \nu^{a,B}(S,y,t)) - v^0(x + C(S,y), y, t)\} = 0. \end{array} \tag{6.59}$$

Knowing the exact formula for $v^0(x,y,t)$, we differentiate it to get:

$$\begin{array}{l} v^0_x(x + \nu^{a,B}(S,y,t)) = \gamma e^{-\gamma(x+\nu^{a,B}(S,y,t)) - \tilde{H}(y,t;T)}, \\ v^0_{xx}(x + \nu^{a,B}(S,y,t)) = -\gamma^2 e^{-\gamma(x+\nu^{a,B}(S,y,t)) - \tilde{H}(y,t;T)}, \\ v^0_{xy}(x + \nu^{a,B}(S,y,t)) = \gamma e^{\gamma(x+\nu^{a,B}(S,y,t)) - \tilde{H}(y,t;T)}\left(-\tilde{H}_y(y,t;T)\right). \end{array} \tag{6.60}$$

Equations (6.21), (6.22) and (6.19) together, imply that

$$\tilde{H}_y(y,t;T)(1-\rho^2) + \frac{f_y(y,t;T)}{f(y,t;T)} = 0, \tag{6.61}$$

and thus

$$v^0_{xy}(x + \nu^{a,B}(S,y,t))(1-\rho^2) = \gamma e^{\gamma(x+\nu^{a,B}(S,y,t)) - \tilde{H}(y,t;T)}\frac{f_y(y,t;T)}{f(y,t;T)}. \tag{6.62}$$

Combining (6.59), (6.60) and (6.62) yields the result of the theorem ■

The next theorem provides another representation of the indifference price, in the form of non-linear functionals $\mathcal{E}_Q(\cdot)$ defined in (6.15). The proof follows closely Proposition 11 in [40] and is omitted here.

$$\nu_t^{a,B}(C;T) = \sup_{t_0 \leq \tau \leq \bar{T}} \left(\mathcal{E}_{\mathbb{Q}^{me}}(C(S_\tau, Y_\tau)) \right). \tag{6.63}$$

The result shown above exists in a similar form in our binomial setting, see section 5.2 theorem 34. In both cases, the continuous and the discrete binomial, the functionals characterizing the indifference value are nonlinear, become linear as risk aversion approaches 0, use the same (minimal entropy) martingale measure, and incorporate the stopping time in the same way, by taking the supremum of all possible buyer's prices obtained for different exercise times. A small difference is that in continuous time the instantaneous correlation coefficient is incorporated into the formula explicitly, but in the discrete time it enter the formula implicitly through the joint distribution of the traded asset and the non-traded factor.

Our results show that the difference between the early exercise indifference prices with Backward and Forward preferences is inherited form their European counterparts. Again, if the Sharpe ratio of the traded asset is constant, then the two variational inequalities (6.52) and (6.37) become the same, as is the case with their European counterparts.

Chapter 7

Extension of the discrete dynamic forward algorithm to partial exercise contracts.

This chapter is devoted to the discussion of the optimal early exercise strategy and the price for situations when an agent us granted several American claims that he could exercise several at a time. In a complete market, the classical no-arbitrage pricing theory suggests that all of the available options have to be exercised at once. When the market is not complete, the additional unhedgeable risk carried by the contract may cause the optimal exercise policy to be partial, even in the absence of transaction costs or trading constraints. For example, [47] use expected utility framework to show that the optimal number of options to be exercised could be only a fraction of the total number held. In incomplete market this problem has been studied extensively in recent years, especially in the context of employee stock options. Among the ones are [11] and [27], who use certainty-equivalent approach to value partial exercise contracts with non-traded assets. There, the opportunity to invest in the market is not taken into consideration when the contracts are being valued. The next step in the development has become including in investor's portfolio a correlated traded asset, to partially hedge the risk, as in [22] or [18]. [22] assume that the Sharpe ratio is constant in continuous time; in discrete time [18] assumes the probability distribution of the traded asset does not depend of the non-traded risk factor, and ξ_t^u, ξ_t^d are constant. We take the analysis even further, by valuing the partially exercised contracts with the non-traded risk factor in a more general, non-reduced model with a dynamically changing agent's utility function.

We assume at any time $0 \leq t \leq T-1$, the buyer would exercise some number of options, add the resulting cash amount to his wealth and then make an investment decision of how many shares of the traded asset to hold over the next period. At time T the buyer only needs to make an exercise decision. As with early exercise, we assume the forward dynamic utility function specified in section 4.1, and that by the terminal time all of the options must be exercised. We also assume the interest rate is zero. We denote the relevant variables as follows:

N - total number of options to be exercised on $[0, \bar{T}]$,

n_t - number of options to be exercised up to and including time t,

β_t - number of options to be exercised at time t,

α_t - number of shares of traded asset to hold on $[t-1, t)$,

X_t - total wealth of the option holder at time t (before exercise takes place at time t)

x - investor's initial wealth at time 0,

C_t - time t intrinsic payoff of each exercised option, $C_t = C(S_t, Y_t)$.

With the assumptions above, investor's wealth accumulates according to

$$X_t = x + \sum_{i=0}^{t-1} C_i \beta_i + \sum_{i=1}^{t} \alpha_t (S_i - S_{i-1}), \tag{7.1}$$

and

$$X_{t+1} = X_t + \beta_t C_t + \alpha_t (S_t - S_{t-1}). \tag{7.2}$$

Non-negative option exercise strategy $\beta = (\beta_0, ..., \beta_{\bar{T}})$ has to satisfy $\sum_{t=0}^{\bar{T}} \beta_i = N$.

Definition 7.1 *We define the value function of the options holder as his maximal expected terminal wealth conditioned on the currently available market information, i.e.*

$$V^C(X_t, n_{t-1}, t; s) = \sup_{\beta_t, ..., \beta_{\bar{T}}} \sup_{\alpha_{t+1}, ..., \alpha_{\bar{T}}} E[U^F_{\bar{T}}(X_{\bar{T}} + \beta_{\bar{T}} C_{\bar{T}}; s) / \mathcal{F}_t] \tag{7.3}$$

Throughout this chapter V^C denotes the partial exercise value function (before V^C was used to denote the early exercise value function). The number of state variables has increased by one compared to the early exercise case, taking into account the number of options exercised before time t.

The definition above in the form of stochastic optimization problem is rather general and we propose a simplified form of the value function, convenient for the derivation of the indifference price.

Theorem 39 (Value function with partial exercise) *If the optimal strategies $\alpha_{t+1},..,\alpha_{\bar{T}}$ and $\beta_t,..,\beta_{\bar{T}}$ in stochastic control problem (7.3) exist for all $t<\bar{T}$*

$$\begin{cases} V^C(X_t,n_{t-1},t;s)=\sup\limits_{0\le\beta_t\le N-n_{t-1}}\sup\limits_{\alpha_{t+1}}E[V^C(X_{t+1},n_{t-1}+\beta_t,t+1;s)], \\ V^C(X_{\bar{T}},n_{\bar{T}-1},\bar{T};s)=U^F_{\bar{T}}(X_{\bar{T}}+N-n_{\bar{T}-1};s). \end{cases} \tag{7.4}$$

Proof. We prove the statement in two parts, by first showing that the left-hand side of (7.3) is at least as big as its right-hand side and then by taking care of the opposite inequality.

We start with arbitrary $\hat{\beta}_t$ and $\hat{\alpha}_{t+1}$ to obtain $\hat{X}_{t+1}=X_t+\hat{\beta}_tC_t+\hat{\alpha}_{t+1}(S_{t+1}-S_t)$. We then use $\bar{\beta}=(\bar{\beta}_{t+1},...,\bar{\beta}_{\bar{T}})$ and $\bar{\alpha}=(\bar{\alpha}_{t+2},...,\bar{\alpha}_{\bar{T}})$ that are optimal in the sense of (7.3) for $V^C(\hat{X}_{t+1},n_{t-1}+\hat{\beta}_t,t+1)$. With the above choice of strategies,

$$\begin{aligned} &E_{\mathbb{P}}[U^F_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}}\beta_{\bar{T}};s)/\mathcal{F}_t]=E_{\mathbb{P}}[E_{\mathbb{P}}[U^F_{\bar{T}}(X_{\bar{T}}+C_{\bar{T}}\beta_t;s)/\mathcal{F}_{t+1}]/\mathcal{F}_t]= \\ &E_{\mathbb{P}}[V^C(\hat{X}_{t+1},n_{t-1}+\beta_t,t+1)/\mathcal{F}_t]. \end{aligned} \tag{7.5}$$

Clearly, $\alpha=(\hat{\alpha}_t,\bar{\alpha})$ and $\beta=(\hat{\beta}_t,\bar{\beta})$ are not necessarily optimal for stochastic control problem (7.3), thus using (7.5)

$$V^C(X_t,n_{t-1},t;s)\ge E_{\mathbb{P}}[V^C(\hat{X}_{t+1},n_{t-1}+\beta_t,t+1;s)/\mathcal{F}_t] \tag{7.6}$$

for any $\hat{X}_{t+1}=X_t+\hat{\beta}_tC_t+\hat{\alpha}_{t+1}(S_{t+1}-S_t)$. Since the choice of $\hat{\beta}_t$ and $\hat{\alpha}_{t+1}$ was arbitrary, the left-hand side of (7.3) is at least as big as its right-hand side. To prove the opposite inequality, assume that $\alpha^\star$ and $\beta^\star$ (and the corresponding $X^\star_s$, $t\le s\le\bar{T}$, constructed via (7.1)) solve (7.3) starting at time t with a wealth level X_t and and a remaining number of options to exercise $N-n_{t-1}$. Then

$$\begin{aligned} &V^C(X_t,n_{t-1},t;s)=E_{\mathbb{P}}[E_{\mathbb{P}}[U^F_{\bar{T}}(X^\star_{\bar{T}}+C_{\bar{T}}\beta_{\bar{T}};s)/\mathcal{F}_{t+1}]/\mathcal{F}_t]= \\ &E_{\mathbb{P}}[E_{\mathbb{P}}[U^F_{\bar{T}}(X^\star_{t+1}+\sum_{i=0}^{\bar{T}-1}C_i\beta^\star_i+\sum_{i=1}^{\bar{T}}\alpha^\star_t(S_i-S_{i-1})+\beta^\star_{\bar{T}}C_{\bar{T}};s)/\mathcal{F}_{t+1}]/\mathcal{F}_t]= \\ &\le E_{\mathbb{P}}[V^C(X^\star_{t+1},n_{t-1}+\beta^\star_t,t+1;s)/\mathcal{F}_t] \\ &\le \sup_{0\le\beta_t\le N-n_{t-1}}\sup_{\alpha_{t+1}}E_{\mathbb{P}}[V^C(X^\star_{t+1},n_{t-1}+\beta^\star_t,t+1;s)/\mathcal{F}_t] \end{aligned} \tag{7.7}$$

■ The next step is to define a family of pricing operators for the case of partial exercise. We start with a one-period operator.

Definition 7.2 *Let $\bar{Z} = (Z_0, ..., Z_N)$ be a vector of $\mathcal{F}_{t+1}$-measurable random variables and $\mathbb{Q}$ be the minimal martingale measure defined earlier in chapter 3. The partial exercise one-period pricing operator $\mathcal{P}^{t,t+1}(\cdot)$ maps $\bar{Z}$ into a vector of $\mathcal{F}_t$-measurable random variables $\bar{\mathcal{Z}} = (\mathcal{Z}_0, ..., \mathcal{Z}_N)$ so that the m-th component of $\bar{\mathcal{Z}}$*

$$\mathcal{Z}_m = \max_{0 \le n \le N-m} C_t n + \mathcal{E}_{\mathbb{Q}}^{t,t+1}(Z_{m+n}), \quad 0 \le m \le N. \tag{7.8}$$

Note that m-th component of the image vector $\bar{\mathcal{Z}}$ only depends on $Z_m, ..., Z_N$ (and does not depend on the first m components of $\bar{Z}$). Using the definition above we recursively construct the multi-period operator $\mathcal{P}_{\mathbb{Q}^{me}}^{t,s}(\cdot)$, $s > t$.

Definition 7.3 *The multi-period pricing operator maps a vector of $\mathcal{F}_s$-measurable random variables $\bar{Z} = (Z_0, ..., Z_N)$ into a vector of $\mathcal{F}_t$-measurable random variables $\bar{\mathcal{Z}} = (\mathcal{Z}_0, ..., \mathcal{Z}_N)$ so that*

$$\mathcal{Z}_m = \mathcal{P}_{\mathbb{Q}^{me}}^{t,t+1}\left(\mathcal{P}_{\mathbb{Q}^{me}}^{t+1,t+2}\left(... \mathcal{P}_{\mathbb{Q}^{me}}^{s-2,s-1}\left(\mathcal{P}_{\mathbb{Q}^{me}}^{s-1,s}\left(\bar{Z}\right)\right) ...\right)\right). \tag{7.9}$$

In other words, the multi-period pricing operator is a composit appropriate number of one-period pricing operators.

Definition 7.4 *We define the buyer's indifference price $p_t(n_{t-1}; C)$ of a partial exercise claim C initiated at t_0 and expiring at $\bar{T}$ as the amount the that satisfies the pricing equation below:*

$$V^C(X_t, n_{t-1}, t; s) = U_t^F(X_t + p_t(n_{t-1}; C); s), \tag{7.10}$$

for all $0 \le n_{t-1} \le N$.

Clearly, the indifference price at time t would depend on n_{t-1}, the number of options exercised before time t. The pricing condition above defines an $N+1$ - dimensional vector of indifference prices $\bar{p}_t(C) = (p_t(0; C), ..., p_t(N; C))$. In what follows we use both, the vector notation and component-wise notation for the indifference price. We are now ready to present the main results of this chapter, analogous to the ones that have already been shown for American claims.

Theorem 40 (Indifference price with partial exercise) (i) *The buyer's indifference price with partial exercise satisfies the recursive formula:*

$$\begin{cases} \bar{p}_{\bar{T}}(C) = C_{\bar{T}} \cdot \bar{N}, \quad \bar{N} = (N, N-1, ..., 0), \\ \bar{p}_t(C) = \mathcal{P}^{t,t+1}_{\mathbb{Q}^{me}}(\bar{p}_{t+1}(C)), \quad t < \bar{T}. \end{cases} \tag{7.11}$$

(ii) *The time t buyer's indifference price is given by:*

$$\begin{cases} \bar{p}_{\bar{T}}(C) = C_{\bar{T}} \cdot \bar{N}, \\ \bar{p}_t(C) = \mathcal{P}^{t,\bar{T}}_{\mathbb{Q}^{me}}(C_{\bar{T}} \cdot \bar{N}), \quad \bar{N} = (N, N-1, ..., 0), t < \bar{T}. \end{cases} \tag{7.12}$$

(iii) *Buyer's indifference price is time-consistent, i.e.*

$$\bar{p}_t(C) = \mathcal{P}^{t,s}_{\mathbb{Q}^{me}}\left(\mathcal{P}^{s,\bar{T}}_{\mathbb{Q}^{me}}\left(C_{\bar{T}} \cdot \bar{N}\right)\right) = \mathcal{P}^{t,s}_{\mathbb{Q}^{me}}\left(\bar{p}_s(C)\right) = \bar{p}_t(\mathcal{P}^{s,\bar{T}}_{\mathbb{Q}^{me}}(C_{\bar{T}} \cdot \bar{N})). \tag{7.13}$$

Proof. (i) We confirm the statement for $t = \bar{T}$, $\bar{T}-1$ and $\bar{T}-2$. The rest could be carried out via induction arguments. At time $\bar{T}$,

$$V^C(X_{\bar{T}}, n_{\bar{T}-1}, \bar{T}) = U^F_{\bar{T}}(X_{\bar{T}} + C_{\bar{T}} \cdot m; s), m = 0, ..., N, \tag{7.14}$$

and the statement follows easily using the definition 7.4.
At time $\bar{T}-1$,

$$\begin{aligned} &V^C(X_{\bar{T}-1}, n_{\bar{T}-2}, \bar{T}-1; s) = \\ &\sup_{0 \le \beta_{\bar{T}-1} \le N - n_{\bar{T}-2}} \sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}[V^C(X_{\bar{T}}, n_{\bar{T}-2} + \beta_{\bar{T}-1}, \bar{T}; s)/\mathcal{F}_{\bar{T}-1}] = \\ &\sup_{0 \le \beta_{\bar{T}-1} \le N - n_{\bar{T}-2}} \sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}[U^F_{\bar{T}}(X_{\bar{T}} + C_{\bar{T}} \cdot (N - n_{\bar{T}-2} - \beta_{\bar{T}-1}); s)/\mathcal{F}_{\bar{T}-1}]. \end{aligned} \tag{7.15}$$

Expression $\sup_{\alpha_{\bar{T}}} E_{\mathbb{P}}[U(X_{\bar{T}} + C_{\bar{T}} \cdot (N - n_{\bar{T}-2} - \beta_{\bar{T}-1}), \bar{T}; s)/\mathcal{F}_{\bar{T}-1}]$ can be viewed as the one-period value function of an investor who holds a European derivative $C_{\bar{T}} \cdot (N - n_{\bar{T}-2} - \beta_{\bar{T}-1})$ expiring at time $\bar{T}$. Therefore, based on the previous results for

European claims,

$$
\begin{aligned}
&V^C(X_{\bar{T}-1}, n_{\bar{T}-2}, \bar{T}-1; s) = \\
&\sup_{0\le\beta_{\bar{T}-1}\le N-n_{\bar{T}-2}} U^F_{\bar{T}-1}(X_{\bar{T}-1} + \mathcal{E}_{\mathbb{Q}}^{\bar{T}-1,\bar{T}}(C_{\bar{T}}\cdot(N-n_{\bar{T}-2}-\beta_{\bar{T}-1})); s) = \\
&U^F_{\bar{T}-1}(X_{\bar{T}-1} + \sup_{0\le\beta_{\bar{T}-1}\le N-n_{\bar{T}-2}} \mathcal{E}_{\mathbb{Q}}^{\bar{T}-1,\bar{T}}(C_{\bar{T}}\cdot(N-n_{\bar{T}-2}-\beta_{\bar{T}-1})); s),
\end{aligned}
\tag{7.16}
$$

Since the forward utility function is monotone in wealth. Comparison of (7.16) evaluated at $X_{\bar{T}-1} - p_{\bar{T}-1}(n_{\bar{T}-2}; C)$ to $U^F_{\bar{T}-1}(X_{\bar{T}-1}; s)$ yields

$$
\begin{aligned}
&p_{\bar{T}-1}(n_{\bar{T}-2}; C) = \sup_{0\le\beta_{\bar{T}-1}\le N-n_{\bar{T}-2}} \mathcal{E}_{\mathbb{Q}}^{\bar{T}-1,\bar{T}}(C_{\bar{T}}\cdot(N-n_{\bar{T}-2}-\beta_{\bar{T}-1})) = \\
&\sup_{0\le n\le N-m} \mathcal{E}_{\mathbb{Q}}^{\bar{T}-1,\bar{T}}(C_{\bar{T}}\cdot(N-(m+n)) = \sup_{0\le n\le N-m} \mathcal{E}_{\mathbb{Q}}^{\bar{T}-1,\bar{T}}(p_{\bar{T}}(m+n; C))).
\end{aligned}
\tag{7.17}
$$

At time $\bar{T}-2$, using the simplified form (7.4) of the partial exercise value function,

$$
\begin{aligned}
&V^C(X_{\bar{T}-2}, n_{\bar{T}-3}, \bar{T}-2; s) = \\
&\sup_{0\le\beta_{\bar{T}-2}\le N-n_{\bar{T}-3}} \sup_{\alpha_{\bar{T}-1}} E_{\mathbb{P}}[V^C(X_{\bar{T}-1}, n_{\bar{T}-3}+\beta_{\bar{T}-2}, \bar{T}-1; s)].
\end{aligned}
\tag{7.18}
$$

Using the definition of $p_{\bar{T}-1}(n; C)$,

$$
\begin{aligned}
&V^C(X_{\bar{T}-1}, n_{\bar{T}-3}+\beta_{\bar{T}-2}, \bar{T}-1) = \\
&U^F_{\bar{T}-1}(X_{\bar{T}-1} + p_{\bar{T}-1}(n_{\bar{T}-3}+\beta_{\bar{T}-2}; C); s).
\end{aligned}
\tag{7.19}
$$

Then,

$$
\begin{aligned}
&V^C(X_{\bar{T}-2}, n_{\bar{T}-3}, \bar{T}-2; s) = \\
&\sup_{0\le\beta_{\bar{T}-2}\le N-n_{\bar{T}-3}} \sup_{\alpha_{\bar{T}-1}} E_{\mathbb{P}}[U^F_{\bar{T}-1}(X_{\bar{T}-1} + p_{\bar{T}-1}(n_{\bar{T}-3}+\beta_{\bar{T}-2}; C); s)/\mathcal{F}_{\bar{T}-2}].
\end{aligned}
\tag{7.20}
$$

The inner supremum in the expression above can be viewed as the time $\bar{T}-2$ one-period value function of an investor who holds the European claim $p_{\bar{T}-1}(n_{\bar{T}-3} + \beta_{\bar{T}-2}; C)$ expiring at $\bar{T}-1$. Then, using results for European payoffs of theorem 14 of section 4.1 again and the definition of the forward dynamic utility U^F of equation

4.2 in section 4.1,

$$
\begin{aligned}
&V^C(X_{\bar{T}-2}, n_{\bar{T}-3}, \bar{T}-2; s) = \\
&\sup_{0\le\beta_{\bar{T}-2}\le N-n_{\bar{T}-3}} U^F_{\bar{T}-1}(X_{\bar{T}-2} + \mathcal{E}^{\bar{T}-2,\bar{T}-1}_{\mathbb{Q}}(p_{\bar{T}-1}(n_{\bar{T}-3}+\beta_{\bar{T}-2}; C); s) = \\
&U^F_{\bar{T}-1}(X_{\bar{T}-2} + \sup_{0\le\beta_{\bar{T}-2}\le N-n_{\bar{T}-3}} \mathcal{E}^{\bar{T}-2,\bar{T}-1}_{\mathbb{Q}}(p_{\bar{T}-1}(n_{\bar{T}-3}+\beta_{\bar{T}-2}; C)); s),
\end{aligned}
\tag{7.21}
$$

and (i) is confirmed for $t = \bar{T}-2$.

(ii) For $t = \bar{T}$ and $t = \bar{T}-1$, statements (i) and (ii) coincide, so it only remains to show (ii) for $t = \bar{T}-2$. We start as in (i) and arrive to (7.18) and (7.20). As shown before,

$$
\begin{aligned}
&p_{\bar{T}-1}(n_{\bar{T}-3}+\beta_{\bar{T}-2}; C) = \\
&\sup_{0\le n\le N-n_{\bar{T}-3}-\beta_{\bar{T}-2}} \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}}(p_{\bar{T}}(n_{\bar{T}-3}+\beta_{\bar{T}-2}+n; C))).
\end{aligned}
\tag{7.22}
$$

Therefore,

$$
\begin{aligned}
&V^C(X_{\bar{T}-2}, n_{\bar{T}-3}, \bar{T}-2; s) = \sup_{0\le\beta_{\bar{T}-2}\le N-n_{\bar{T}-3}} \sup_{\alpha_{\bar{T}-1}} E_{\mathbb{P}}[U^F_{\bar{T}-1}(X_{\bar{T}-1}+ \\
&\sup_{0\le n\le N-n_{\bar{T}-3}-\beta_{\bar{T}-2}} \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}}(p_{\bar{T}}(n_{\bar{T}-3}+\beta_{\bar{T}-2}+n; C)); s)] \\
&= \sup_{0\le\beta_{\bar{T}-2}\le N-n_{\bar{T}-3}} U^F_{\bar{T}-1}(X_{\bar{T}-2}+ \\
&\mathcal{E}^{\bar{T}-2,\bar{T}-1}_{\mathbb{Q}}(\sup_{0\le n\le N-n_{\bar{T}-3}-\beta_{\bar{T}-2}} \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}}(p_{\bar{T}}(n_{\bar{T}-3}+\beta_{\bar{T}-2}+n; C))); s) \\
&= U^F_{\bar{T}-2}(X_{\bar{T}-2} + \sup_{0\le\beta_{\bar{T}-2}\le N-n_{\bar{T}-3}} \\
&\mathcal{E}^{\bar{T}-2,\bar{T}-1}_{\mathbb{Q}}(\sup_{0\le n\le N-n_{\bar{T}-3}-\beta_{\bar{T}-2}} \mathcal{E}^{\bar{T}-1,\bar{T}}_{\mathbb{Q}}(p_{\bar{T}}(n_{\bar{T}-3}+\beta_{\bar{T}-2}+n; C)))); s) \\
&= U^F_{\bar{T}-2}(X_{\bar{T}-2} + \mathcal{Z}_{n_{\bar{T}-3}}; s),
\end{aligned}
\tag{7.23}
$$

where $\mathcal{Z}_{n_{\bar{T}-3}}$ is the $n_{\bar{T}-3}$-th component of $\mathcal{P}^{\bar{T}-2,\bar{T}}_{\mathbb{Q}^{me}}(\bar{p}_{\bar{T}}(C))$.

$V^C(X_{\bar{T}-2} + p_{\bar{T}-2}(n_{\bar{T}-3}; C), n_{\bar{T}-3}, \bar{T}-2; s)$ can be computed similarly and, compared to $U^F_{\bar{T}-2}(X_{\bar{T}-2}; s)$, yields

$$
\bar{p}_{\bar{T}-2}(C) = \mathcal{P}^{\bar{T}-2,\bar{T}}_{\mathbb{Q}^{me}}(\bar{p}_{\bar{T}}(C)).
\tag{7.24}
$$

(iii) The proof of the last part follows from part (ii) and the definition of operators $\mathcal{P}^{t,s}_{\mathbb{Q}^{me}}(\cdot)$ ■

The formula for the partial exercise operator (7.8) translates into choosing the number of options to be exercised this period in such a way that indifference value of the next periods partial exercise price is maximized. If the investor holds only one unit of the American contract, and the only two alternatives available for the investor each period are to exercise one unit of options or none, then the maximum in formula (7.8) reduces to the maximum over $C(S_t, Y_t)$ and the next periods indifference value of one American contract. That exactly the early exercise algorithm we derived for American claims earlier, meaning that the current algorithm is an extension of the one developed herein for American contracts.

The one-period pricing operators, defined in chapter 2 equation (2.9) under the minimal martingale measure, create a foundation for the early exercise and partial exercise pricing algorithms. As a result, all of the natural properties of the European indifference prices are transferred onto pricing of contracts with partial exercise. One could rigorously show that the partial exercise algorithm yields prices that are monotone in risk aversion(i.e., the more risk averse investor would agree pay a lower price). Also, as before, the nonlinearities in the pricing would go away if the risk aversion level becomes very small, or if the non-traded risk factor is in "perfect correlation" with the traded stock.

Chapter 8

Examples of numerical implementation.

This chapter is devoted to numerical implementation of the derived algorithm for European and American derivatives. We consider several examples with non-traded assets that show dependence of the indifference price on correlation between the traded and the non-traded asset, dependence on risk aversion, demonstrate the difference between the prices for the American and European contracts of the same length, and show the difference between prices with backward and forward preferences.

Our first model example is the discrete time analog of the continuous time model consisting of the traded asset S and non-traded stochastic factor Y, satisfying their corresponding SDEs as:

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t^1, \\ dY_t = bY_t dt + aY_t dW_t^2, \end{cases} \tag{8.1}$$

with dW_t^1 and dW_t^2 being the two Brownian motions correlated with a coefficient ρ. In this example, μ, σ, b, a and ρ are constant. According to the chosen framework we construct a binomial tree with the following parameters:

$$
\begin{aligned}
\xi_t^u &= 1 + \mu dt + \sigma\sqrt{dt}, \\
\xi_t^d &= 1 + \mu dt - \sigma\sqrt{dt}, \\
\eta_t^u &= 1 + bdt + a\sqrt{dt}, \\
\eta_t^d &= 1 + bdt - a\sqrt{dt}
\end{aligned}
\tag{8.2}
$$

with $-\sigma < \mu\sqrt{dt} < \sigma$, $-a < b\sqrt{dt} < a$ and $dt = T/N$. Also, for all $0 <= t <= T$,

$$
\begin{aligned}
&P(S_t = S_t^u/\mathcal{F}_{t-1}) = 0.5, \\
&P(Y_t = Y_t^u/\mathcal{F}_{t-1} \vee \mathcal{F}_t^S) = \begin{cases} \dfrac{1+\rho}{2}, S_t = S_t^u, \\ \dfrac{1-\rho}{2}, S_t = S_t^d. \end{cases}
\end{aligned}
\tag{8.3}
$$

Note that with such a set of parameters

$$
\begin{aligned}
&E_P[\xi_t/\mathcal{F}_{t-1}] = 1 + \mu, \\
&E_P[\eta_t/\mathcal{F}_{t-1}] = 1 + b + a\sqrt{dt} \cdot (2P(Y_t = Y_t^u/\mathcal{F}_{t-1}) - 1) = \\
&1 + b + a\sqrt{dt} \cdot (2 \cdot (P(Y_t = Y_t^u/\mathcal{F}_{t-1} \vee S_t = S_t^u) \cdot P(S_t = S_t^u/\mathcal{F}_{t-1}) + \\
&P(Y_t = Y_t^u/\mathcal{F}_{t-1} \vee S_t = S_t^d) \cdot P(S_t = S_t^d/\mathcal{F}_{t-1})) - 1) = \\
&1 + b + a\sqrt{dt} \cdot (2 \cdot (\frac{1+\rho}{2} \cdot P(S_t = S_t^u/\mathcal{F}_{t-1}) + \\
&\frac{1-\rho}{2} \cdot (1 - P(S_t = S_t^u/\mathcal{F}_{t-1}))) - 1) = \\
&1 + b + a\sqrt{dt}\rho(2P(S_t = S_t^u/\mathcal{F}_{t-1}) - 1) = 1 + b.
\end{aligned}
\tag{8.4}
$$

Thus,

$$
\begin{aligned}
&E_P[\Delta S_t/\mathcal{F}_{t-1}] = S_{t-1}\mu dt, \\
&E_P[\Delta Y_t/\mathcal{F}_{t-1}] = Y_{t-1} b dt, \\
&Var[\Delta S_t/\mathcal{F}_{t-1}] = S_{t-1}^2 \sigma^2 dt, \\
&Var[\Delta Y_t/\mathcal{F}_{t-1}] = Y_{t-1}^2 a^2 dt.
\end{aligned}
\tag{8.5}
$$

Note also that

$$
\begin{aligned}
&Cov[\Delta S_t, \Delta Y_t/\mathcal{F}_{t-1}] = S_{t-1} Y_{t-1} a\sigma dt \\
&(P(S_t = S_t^u, Y_t = Y_t^u/\mathcal{F}_{t-1}) + P(S_t = S_t^d, Y_t = Y_t^d/\mathcal{F}_{t-1}) - \\
&P(S_t = S_t^u, Y_t = Y_t^d/\mathcal{F}_{t-1}) - P(S_t = S_t^d, Y_t = Y_t^u/\mathcal{F}_{t-1})) = \\
&S_{t-1} Y_{t-1} a\sigma dt \left(\frac{1+\rho}{2} \cdot 0.5 + \left(1 - \frac{1-\rho}{2}\right) \cdot 0.5 - \right. \\
&\left. \left(1 - \frac{1+\rho}{2}\right) \cdot 0.5 - \frac{1-\rho}{2} \cdot 0.5\right) = S_{t-1} Y_{t-1} a\sigma dt\rho,
\end{aligned}
\tag{8.6}
$$

and therefore

$$Cor[\Delta S_t, \Delta Y_t / \mathcal{F}_{t-1}] = \rho. \tag{8.7}$$

In this example, the historical probability distribution of the next period's increase in the level of the traded asset S, given the currently available market information, is constant throughout time and equals 0.5. As confirmed by the theoretical results of the previous chapters, the minimal martingale measure and the minimal entropy measure then coincide. In this example we will be referring to that measure as the minimal martingale measure $\mathbb{Q}^{mm}$, that satisfies:

$$\mathbb{Q}^{mm}(S_t = S_t^u / \mathcal{F}_{t-1}) = \frac{1-\xi_t^d}{\xi_t^u - \xi_t^d} = \frac{1}{2} - \frac{\mu\sqrt{dt}}{2\sigma}. \tag{8.8}$$

For the non-traded risk factor

$$\begin{aligned}
&\mathbb{Q}^{mm}(Y_t = Y_t^u / \mathcal{F}_{t-1}) = \\
&\mathbb{Q}^{mm}(Y_t = Y_t^u / \mathcal{F}_{t-1} \vee S = S_t^u) \cdot \mathbb{Q}^{mm}(S_t = S_t^u / \mathcal{F}_{t-1}) + \\
&\mathbb{Q}^{mm}(Y_t = Y_t^u / \mathcal{F}_{t-1} \vee S = S_t^d) \cdot \mathbb{Q}^{mm}(S_t = S_t^d / \mathcal{F}_{t-1}) = \\
&P(Y_t = Y_t^u / \mathcal{F}_{t-1} \vee S = S_t^u) \cdot \mathbb{Q}^{mm}(S_t = S_t^u / \mathcal{F}_{t-1}) + \\
&P(Y_t = Y_t^u / \mathcal{F}_{t-1} \vee S = S_t^d) \cdot \mathbb{Q}^{mm}(S_t = S_t^d / \mathcal{F}_{t-1}) = \\
&\frac{1+\rho}{2} \cdot \left(\frac{1}{2} - \frac{\mu\sqrt{dt}}{2\sigma}\right) + \frac{1-\rho}{2} \cdot \left(1 - \left(\frac{1}{2} - \frac{\mu\sqrt{dt}}{2\sigma}\right)\right) = \\
&\frac{1}{2} - \rho\frac{\mu\sqrt{dt}}{2\sigma}.
\end{aligned} \tag{8.9}$$

The graphs shown in Figure 8.1 is a plot of the indifference price of the European basket call option. We used $\mu = b = 0$, $\sigma = 0.2$, $a = 0.4$, $\gamma = 0.5$ and 0.2, ρ varying from -1 to 1 with a step size of 0.2. The call matures in 1 month and we used 100 time steps to make the graph. The graph shows both, the writer's price curves, the buyer's price curves for two different risk aversion values, and the "risk-neutral" price(i.e., the expectation of the contracts payoff under the minimal martingale measure). The highest curve on the graph corresponds to the writer's price when γ equals to 0.5. The lowest curve is the buyer's price for $\gamma = 0.5$. The middle curve is the "risk-neutral" price, and the two curves immediately above and below the "risk-neutral" curve are the writer's and buyer's prices for $\gamma = 0.2$. The graph shows that prices are monotone with respect to risk aversion, and show a dependence on the correlation between the asset and the non-traded factor. The dependence on the correlation is non-linear, and not always monotone, as for example with $\gamma = 0.5$. As confirmed theoretically, the dependence on γ is monotone, and the price is higher for a more risk averse writer, the opposite being true for the buyer. Also note that independently γ, the prices coincide when $\rho = 1$ or $\rho = -1$. In the case of perfect correlation or perfect anti-correlation, the distribution of the non-traded factor given the value of the traded stock is known with certainty. For those values of correlation, the nonlinearity of the indifference goes away independently of what the level of risk aversion is. Thus, as confirmed in Figure 8.1, at $\rho = 1$ or $\rho = -1$, indifference prices are the same for all values of γ.

Figure 8.2 confirms results computed with binomial model by comparing them against results obtained for the original continuous model (8.1). The green (upper) curve represents the output obtained with our binomial algorithm for the same parameters as in Figure 8.1 with $\gamma = 0.5$ for the writer's price. The blue (lower) curve represents the indifference value of the same contract computed as a solution of the nonlinear indifference price PDE, using the method developed by [44]. The method uses an explicit finite difference scheme, with the spacial step size of 0.1 in both S and Y, and time step size of 0.0002. For more details on the implementation of indifference priced via solutions of PDEs we refer the reader to [44]. As seen from the graph, the two outputs are the same at least up to two significant digits. Thus the two methods could be used to cross-check each other's results.

The graph in Figure 8.3 shows a family of curves that represent prices of American put option on a non-traded risk factor, maturing at T with a strike price K, shown as a function of initial risk factor value Y_0. We used the following data to make the graph: $\mu = \sigma = 1$, $a = b = 1$, $T = 1$, $N = 100$, $K = 1$, $\gamma = 0.1$. The parameters used are the same as in [44]. The upper most curve on the graph corresponds to $\rho = 1$ and the lower most curve to $\rho = -1$. Clearly, the value of the put option decreases in Y_0, as expected. Also, the price of a put option is monotone in the correlation between the traded stock and the risk factor. As ρ increases from -1 to 1, the conditional on time t probability of Y going up (under the minimal entropy measure $\mathbb{Q}^{mm}$) decreases, as computed via equation (8.9):

$$\mathbb{Q}^{mm}(Y_t = Y_t^u / \mathcal{F}_{t-1}) = \frac{1}{2} - \rho \frac{\mu\sqrt{dt}}{2\sigma}. \tag{8.10}$$

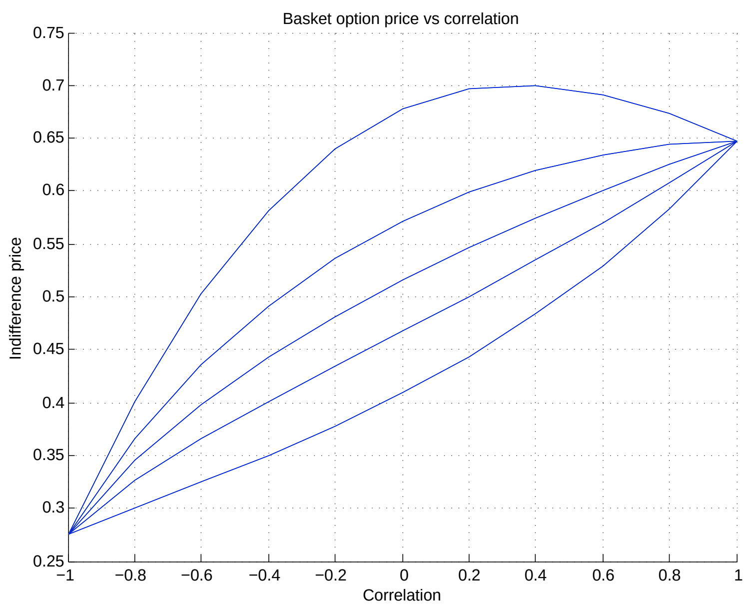

Figure 8.1: Dependence of Indifference price on correlation and risk aversion

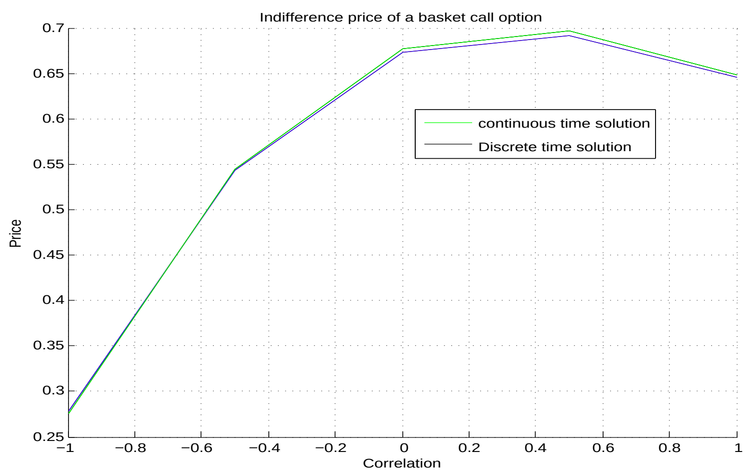

Figure 8.2: Binomial and Continuous implementation for a basket call option

Thus, when ρ increases, the put option on the non-traded risk factor gains value, which is what we see in the graph of Figure 8.3. In addition, shown in the same graph is initial intrinsic value $(Y_0-K)^+$ of the put, as a function of Y_0 (solid black line). As it should be with an American option, the price of the option is always greater than its intrinsic value. Figure 8.3 indicates position of the free boundary. According to the graph, in-the-money put may be exercised if the correlation between the stock and the factor is small enough, especially if negative.

Figures 8.4-8.6 shows selected curves for both European and American options together for direct comparison. The graph of Figure 8.4 corresponds to $\rho = 1$. In that case the probability of Y going up reaches its maximum value and there is no incentive for the buyer to exercise the option early. Thus, the prices of European and American options coincide. The graph in Figure 8.5 shows European and American options for $\rho = 0$. The probability of Y going up is now smaller than in the upper graph of the Figure 8.4 and thus there are more incentives to exercise the option early, which results in a bigger difference in between the European and the American prices. For the graph in Figure 8.6, the correlation is set to -1 and the difference between the American and the European option prices becomes even more pronounced.

In the previous model example (as given by 8.2 and 8.3), dynamics of the traded asses were such that the probability distribution of the next period's value of the asset is not affected by the path of the non-traded factor, and constant ξ_t^u, ξ_t^d, that is the assumption 10 of chapter 3 holds. As a result, there is no difference between the indifference prices obtained with either forward or backward dynamic preferences. To provide an example where the two are in fact different, we slightly modify the previous reduced model example by re-defining the historical probability of the upward traded asset movement to be

$$\mathbb{P}(S_{t+1} = S_{t+1}^u / \mathcal{F}_t) = 0.5 \exp((Y_t - Y_0)dt). \tag{8.11}$$

That way, if $Y_t > Y_0$, the probability of S going up would be slightly bigger then 0.5, and slightly smaller then 0.5 otherwise. This example may not have the continuous time limit as dt approaches 0, but it is well suited for our purpose of demonstrating the difference between the Forward and the Backward prices. With this example, we confirm that our algorithm produces forward price that is greater then the backward.

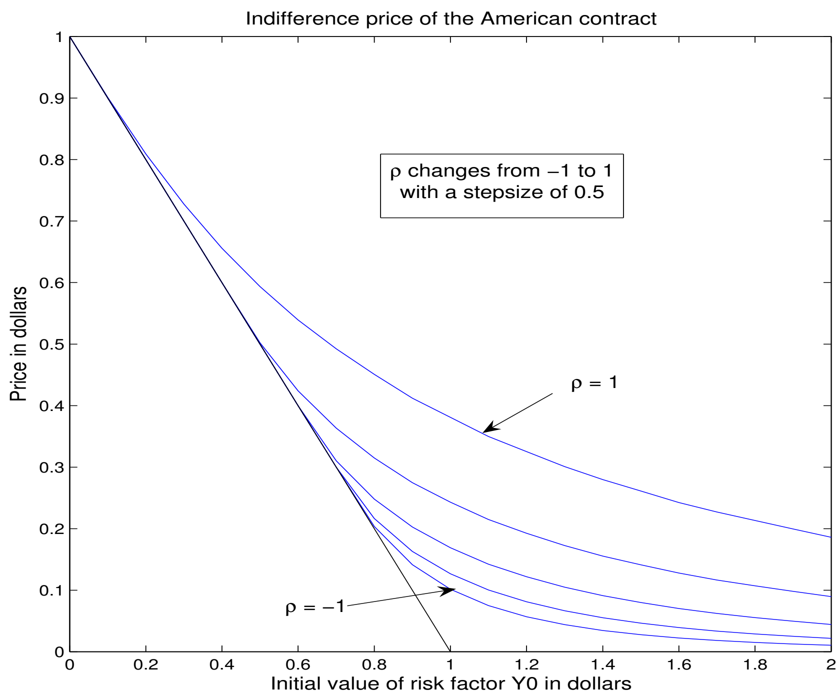

Figure 8.3: Position of the early exercise boundary in a survey over correlation

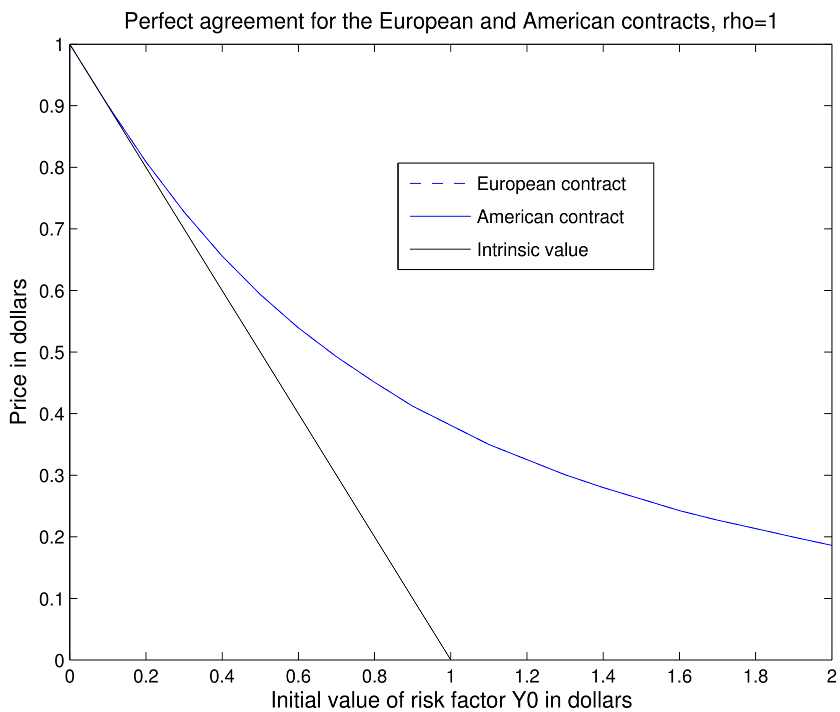

Figure 8.4: European vs American equal maturity put prices, dependence on correlation

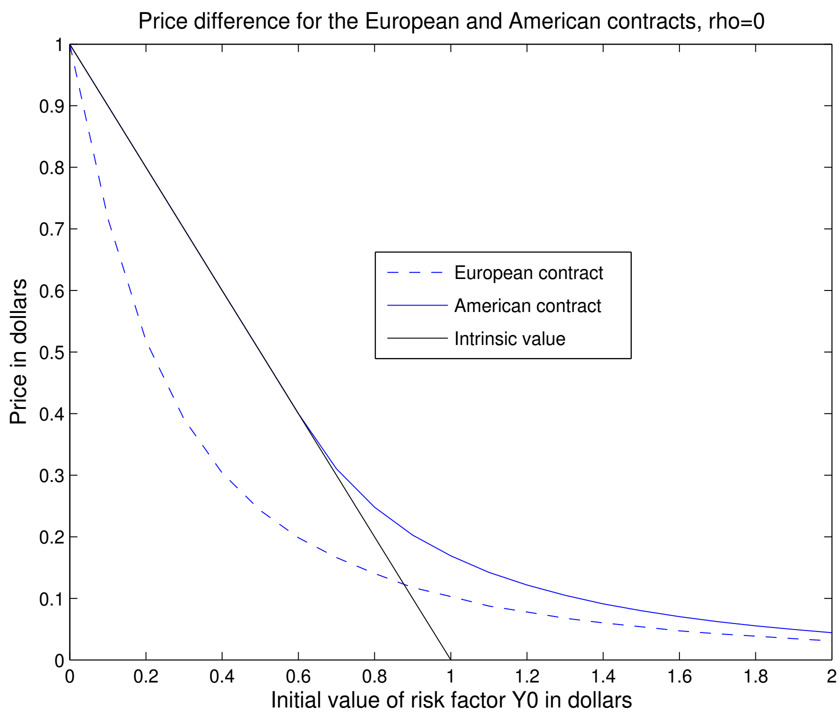

Figure 8.5: European vs American equal maturity put prices, dependence on correlation

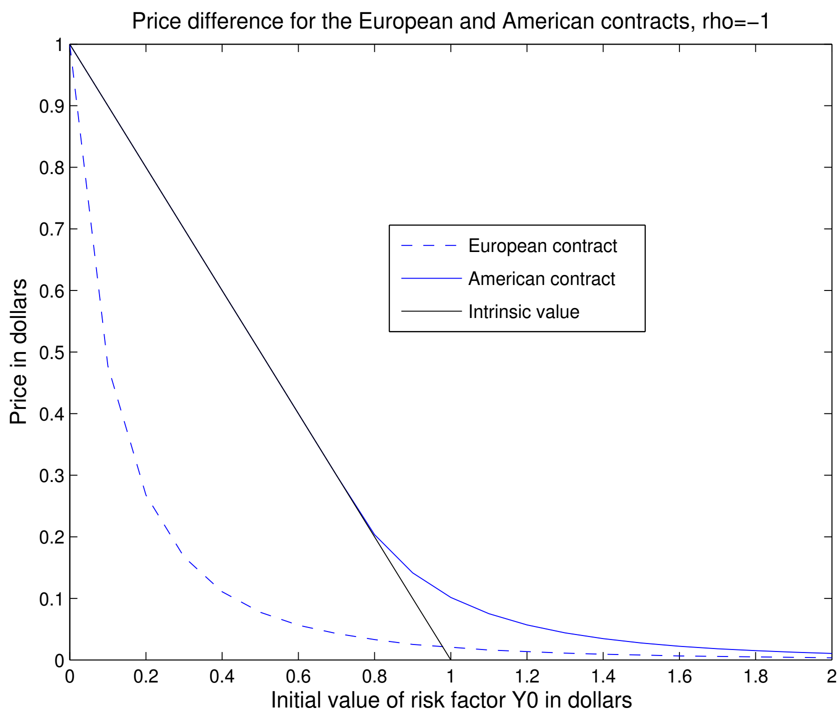

Figure 8.6: European vs American equal maturity put prices, dependence on correlation

We again use basket call option with $a = 0.4$, $\gamma = 0.75$, $\mu = 0.05$, $S_0 = Y_0 = 10$, $K = 20$, $\sigma = 0.2$, $\rho = 0.5$. Maturity of the call option is fixed at 1 month and the time step is $\frac{1}{12 \cdot 60}$. We used the same discretization for ξ_t and η_t as in equation 8.2, the same $\mathbb{P}(Y_t = Y_t^u / \mathcal{F}_{t-1}) = 0.5$, but $\mathbb{P}(S_{t+1} = S_{t+1}^u / \mathcal{F}_t)$ is not 0.5 but instead given by equation (8.11).

Figure 8.7 shows the forward and the backward priced for the buyer(lower two curves) and the writer(upper two curves) of the fixed, 1 month maturity basket call option with parameters as described above and investment horizon of 3 month. We plot the prices as functions of parameter ρ. The forward price turned out to be greater then the backward, for both the buyer and the writer, but we believe this might not be generally the case. Also, in this case the difference is more pronounced for the buyer rather then for the writer.

To make Figure 8.8, we kept maturity of the call option fixed at 1 month, but let the trading horizon (that is, the normalization point for the backward preferences) vary form 1 month to 2 month. As a result, the calculated forward price did not show any changes, unlike the "horizon-dependent" backward price, that was decreasing, the difference between the two getting bigger as the investment horizon increased. Our numerical surveys indicate that the difference between the backward and the forward price is small for shorter maturity contracts; it gets larger when the maturity of the option and the end of investment horizon become more apart. Whether for the writer, then the opposite would hold for the buyer, but it does not seem to necessarily be the case.

The next example presented here illustrates the partial exercise algorithm developed in the earlier chapters. Investor holds a total of 50 put options on a non-traded risk factor. He should exercise all of the options by the maturity date of the contract, which is a quarter of a year away from now. Before the maturity date, there are 20 available equally spaced exercise periods, during which the investor could exercise from 0 to 50 options. This example mostly uses the same discretization as described above with the following parameters: $a = 0.2$, $b = 0.05$, $\gamma = 0.5$, $\mu = 0.05$, $S_0 = 10$, $\sigma = 0.2$, $\rho = 0.5$, $K = 10$. The discretization of the historical probability distribution is slightly different compared to the previous example. Here,

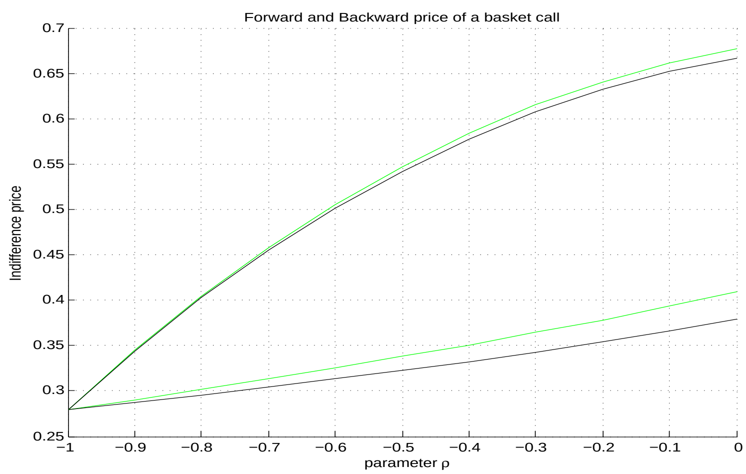

Figure 8.7: Backward and Forward prices for basket call option

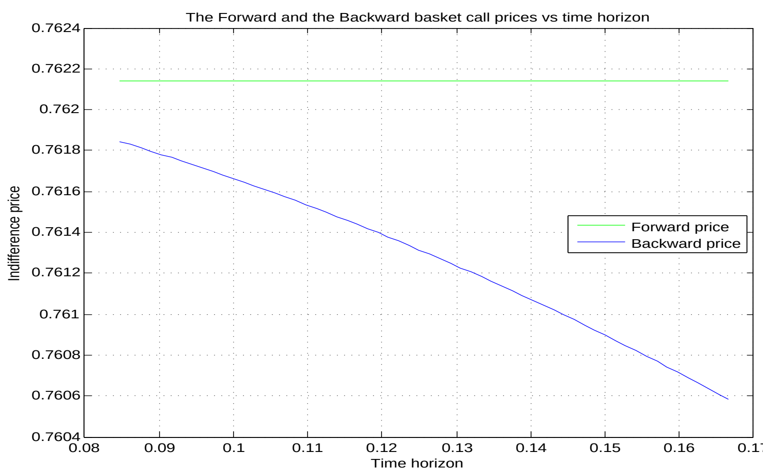

Figure 8.8: Backward and Forward prices vs time horizon

the conditional distribution of Y_{t+1} given S_{t+1} is homogeneous in time and such that

$$\mathbb{P}(Y_{t+1} = Y^u_{t+1}/\mathcal{F}_t \vee S_{t+1}) = \begin{cases} \dfrac{(1-\rho)}{2}, \text{ for } S_{t+1} > S_t, \\ \dfrac{(1+\rho)}{2}, \text{ for } S_{t+1} \le S_t \end{cases} \tag{8.12}$$

and

$$\mathbb{P}(S_{t+1} = S^u_{t+1}/\mathcal{F}_t) = \begin{cases} 0.55, \text{ for } Y_{t+1} > Y_0, \\ 0.45, \text{ for } Y_{t+1} \le Y_0 \end{cases} \tag{8.13}$$

With such choice of the historical distribution, the distribution of the traded asset is affected by the non-traded factor. Therefore, the backward and forward prices would not coincide.

Figures 8.9, 8.10 and 8.11 present the forward partial exercise price, the optimal number of options to hold and the optimal investment policy, all as functions of the initial value of the non-traded risk factor $Y0$ and the number of time periods left until maturity of the contract. Y_0 changes from 9.9 to 10.1. The graphs show that the price decreases with the current non-traded factor value Y_0 and increases with time to maturity.

In a complete market, all of the contracts held should be exercised at once. In this example, contract's payoff depends on the non-traded risk factor. As a result, the optimal exercise policy is indeed partial, as shown in figure 8.10. The optimal number of options to hold increases with Y_0, since a higher initial value of Y_0 reduces the payoff of put option units exercised immediately. The optimal number of options to hold also decreases with time to maturity. The optional investment policy indicates holding less shares of traded stock for a higher level of the non-traded risk factor, and also less shares of the traded asset for longer contract's maturity.

One of the applications for non-reduced discrete binomial model developed in the earlier chapters would be to the models stochastic volatility, such as Heston model, originated in [23], in which the traded asset and the non-traded stochastic process satisfy:

$$\begin{cases} dS_t = \mu S_t dt + S_t \sqrt{Y_t} dW^1_t, \\ dY_t = \kappa(\bar{\sigma} - Y_t)dt + \sqrt{Y_t} dW^2_t, \end{cases} \tag{8.14}$$

where dW^1_t and dW^2_t are the two Brownian motions correlated with an instantaneous correlation coefficient ρ. One could discretize SDEs for S and Y in a way similar to the earlier examples, but in that case parameters ξ^u_t and ξ^d_t would depend on the

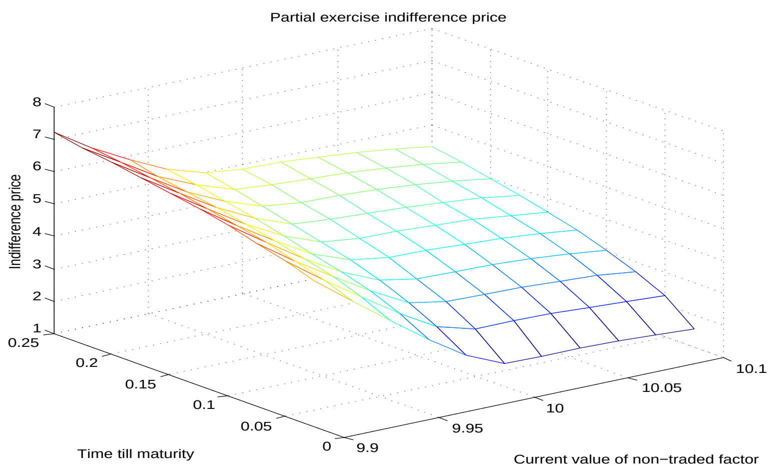

Figure 8.9: Partial exercise price for 50 put option on non-traded asset

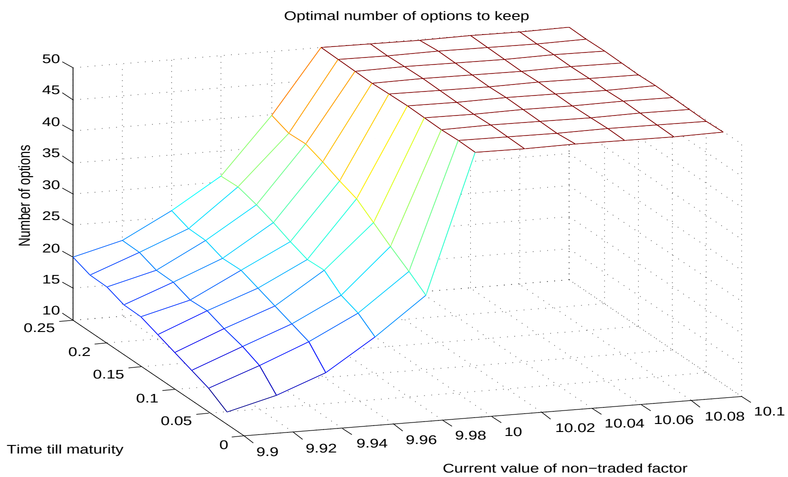

Figure 8.10: Optimal number of options to hold

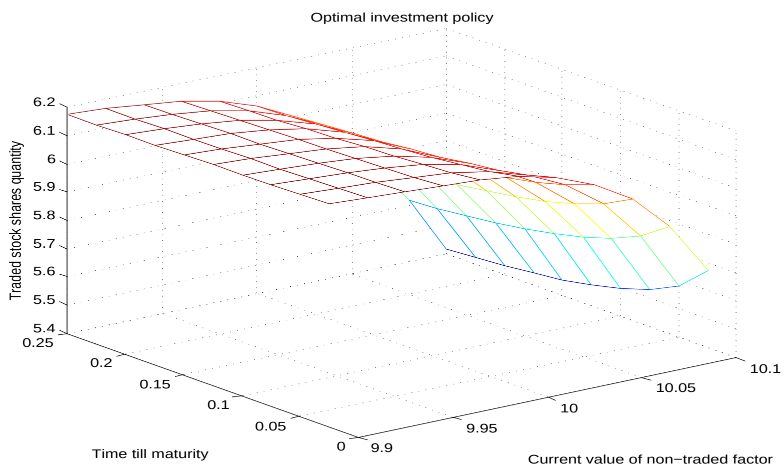

Figure 8.11: Optimal investment policy with partial exercise

whole path from time 0 up to time t of the stochastic factor and the corresponding binomial tree would not recombine. Working with non-recombining binomial trees requires substantial computational resources and is not effective. A number of ways have been suggested in the literature to reduce the problem to constructing binomial trees that would recombine. A classical approach is to transform the stochastic equation above into a simpler one, using a different pair of stochastic variables, as in [24] for example. The transformation maps the variable S_t into $\tilde{S}_t = f(S_t, Y_t)$. Since function f in general, depends on the value of the non-traded factor, the new variable $\tilde{S}_t$ may not be perfectly replicable. With the transformation approach, our algorithm could not be applied to the new state variable $\tilde{S}$, implying that an efficient numerical application of the algorithm to stochastic volatility models is yet to be developed.

In continuous time, a stochastic volatility model is numerically implemented using finite difference techniques for partial differential equations. Below we provide an example of implementation of the early exercise indifference price with forward preferences in Heston stochastic volatility model shown above. We have used the following parameters: $K = 10$, $Y_0 = 0.25$, $\bar{\sigma} = 0.25$, $\mu = 0.05$, $a = 0.5$, $\kappa = 1.0$, $\gamma = 0.5$, $\rho = -0.5$, $\bar{T} = T = 1.2$ years. The graph in figure 8.12 shows the early exercise indifference price for the buyer of the put option on the traded asset S with a strike K, as a function of the initial traded asset value S_0 varying from 8 to 12 with a step size of 1. the early exercise price is greater than the intrinsic payoff, and in decreasing in the initial traded asset value S_0. Figure 8.12 also shows the option's intrinsic payoff, as a function of the initial traded asset value S_0.

The partial differential equation that the indifference price with the forward preferences, $\nu^{a,F}$ satisfies has been derived in chapter (Reference) equation 6.37:

$$\begin{cases} \min\{-\nu_t^{a,F} - \mathcal{L}^{mm}\nu^{a,F} + \frac{1}{2}\gamma(1-\rho^2)a(y)^2(\nu_y^{a,F})^2, \nu^{a,F} - C(S,y)\} = 0, \\ \nu^{a,F}(\bar{T}) = \max(S-K, 0). \end{cases} \tag{8.15}$$

with

$$\mathcal{L}^{mm} = \frac{1}{2}S^2 y\frac{\partial^2}{\partial S^2} + \rho S a y\frac{\partial^2}{\partial S \partial y} + \frac{1}{2}a^2 y\frac{\partial^2}{\partial y^2} + \left(\kappa(\theta - y) - \rho\mu a\right)\frac{\partial}{\partial y} \tag{8.16}$$

We have made a change of variables $S = S_0 \cdot e^u$, $y = Y_0 \cdot e^z$, with u and z varying from -3 to 3 with a step size of 0.05.

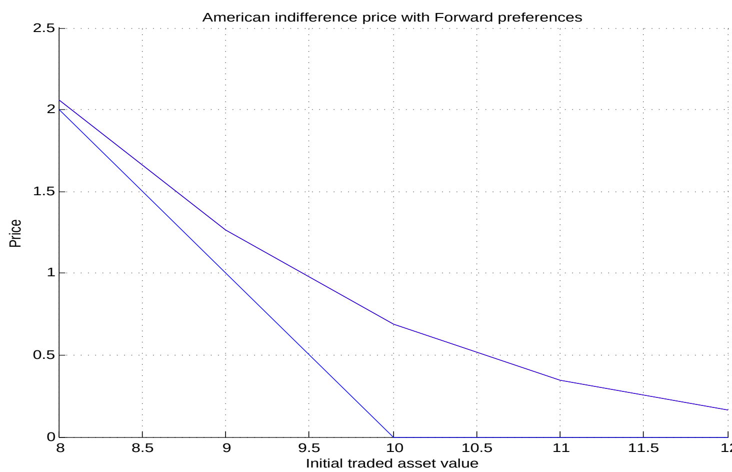

Figure 8.12: Forward indifference price for an American put

With the above parameter values we have been able to see that the early exercise price almost coincides with the price of the corresponding European option with maturity at $\bar{T}$. Therefore, with the model chosen and the parameters chosen, the call on the traded asset should never be exercised. We have discussed in section 5.3, that in the non-reduced model, as the stochastic volatility model considered above, there may be situations when a call on the traded asset S would be exercised early, but with the above model parameters that is not the case. That may be an indiaction that the particular example used is numerically very close to satisfying a reduced model condition. We also confirmed the above by computing the backward and the forward prices directly. We have obtained, for example, the difference on order of 0.1 and 0.2 percent for maturities of 0.9 and 1.2 years correspondingly. Those differences between the backward and the forward prices were not sufficient to be above the error of the numerical implementation. However, there is a trend of the difference between the two becoming larger as the length of the contract increases. The finite difference method we used is an explicit finite difference method of [44]. It has been specifically developed for early exercise contracts and satisfies the necessary conditions for convergence. As for any explicit scheme, the time step size has to be restricted in order for the numerical scheme to converge. To compute the price for a 1.2 year contract we had to use $6 \cdot 10^3$ steps with a time step of $2 \cdot 10^{-4}$. Clearly, in order to have the contract length of a few years, too many time steps are needed. We conclude that, to see a significant difference between the backward and the forward price of a call option in a stochastic volatility model, a longer contract should be considered, possibly with another finite difference scheme. To avoid time step size restrictions, an implicit finite difference scheme may be chosen, together with a penalty method to incorporate the early exercise feature.

Chapter 9

Conclusion and future work.

This thesis contributes to the vast literature on the valuation of derivative contracts in incomplete markets. We work with a special type of market incompleteness, in which the contract's value is affected by the non-traded risk factor. The latter can affect the value of the contract in two distinct ways, the first of which is through the functional form of the payoff; the second way is that the non-traded factor may explicitly enter the the dynamics of the traded asset. The first way is subtle and usually yields a simpler valuation procedure. The second way is more fundamental and yields more involved pricing. The majority of explicit pricing results derived so far have are for the first type of dependency. In this work we focused on the second, more fundamental type of dependence, and yet were able to preserve the explicit character and tractability of our results.

The discrete time model project presented here was started shortly after the results of [2] and [41] became available. The discrete model suggested in the latter reference served as a starting point for our work. We extended the model so that we could separate the two types of dependence on the non-traded risk factor described above. Our model is also attractive in that its computational implementation uses a well-known and easily implementable method of computing the value of the option on a binomial tree, originated in [8].

The original model of [41] is a reduced model. As a result, in that model the minimal entropy measure and the minimal martingale measure are the same, and the minimal entropy measure has a simpler characterization. In that model the local entropy terms (that are the discrete time analog of the Sharpe ratio) are not affected by the non-traded risk factor and the aggregate minimal entropy accumulates using the linear operator (the expectation under the minimal entropy measure). In our work, the reduced model assumption has been forfeited, opening the way for the new characterization of the minimal entropy martingale measure and the new representation for the aggregate entropy. It turns out that in the non-reduced model, the minimal entropy measure has a more complex representation and the aggregate entropy is a nonlinear functional of the sum of the local entropy terms. Also, the two measures, the minimal martingale and the minimal entropy, are different. Explicit characterization of two measures has helped us to draw a clear difference between them, and has had a significant impact on the pricing results. The new characterizations have also helped to reconcile our discrete time model with the continuous time model developed in [42] and [54].

Another common assumption in the literature considering the valuation of contracts in an incomplete market using utility maximization is the classical static exponential utility function, which is usually fixed at contract's expiration date. This is a somewhat restrictive assumption since it imposes a time constraint on the maturities of contracts being priced. Also, it creates an artificial mispricing, since the two differently chosen investment horizons yield different prices for the same contract. In an attempt to avoid the above difficulty, [42] introduced the new concept of the forward dynamic utility in continuous time. The forward utility allows the investment horizon and contract's expiration to be different, and does not impose any time constraints on the maturities of contracts. Motivated by the new concept, we developed the multi-period pricing algorithm with the forward dynamic utility in discrete time. The indifference valuation has long been associated with the minimal entropy measure, and the construction of [42] has made an unexpected turn by showing that one can construct the utility process in such a way that the associated pricing measure is the minimal martingale measure, and not the minimal entropy one. That is also the case with our discrete time model, i.e., one can use the minimal martingale measure as the pricing measure and the indifference pricing framework combined, with all their attractive properties.

Another way to extend the classical static exponential utility to a dynamic one is to use the so-called plain investment value function discussed in chapter 2. As [42] suggest and as we confirmed in the discrete time model, the dynamic utility chosen in the above way satisfies the self-generation property and thus is consistent across different investment horizons. The above utility has been termed the backward dynamic utility process. In this work we constructed its discrete version in a more general, non-reduced model.

Working with both forward and backward dynamic utilities, we developed a discrete time algorithm for valuing, first European, and then American, and partial exercise contracts. The forward and backward valuation algorithms are similar in that both are iterative and compute the prices backward in time, starting at the contract's expiration. They have the same structure, in that at each valuation step there are two sub-steps: the first one that prices the non-hedgeable risk with conditional certainty equivalents and the second prices linearly the remaining hedgeable risk. When the market becomes complete, the first sub-step does not come into effect and both forward and backward prices reduce to the arbitrage-free price. With both utilities, the form of the nonlinear pricing functional is time-invariant, and dependence on the normalization points or the end of the investment horizon can only enter through the measure used for pricing.

There are, however, some important differences. The most immediate difference is that the algorithms use different pricing measures, the minimal martingale measure for the forward and the minimal entropy measure for the backward. In this work we provided explicit characterization for both of these measures. We have shown that the property of preserving the historical probability distribution of the non-traded risk factor, given the next period's value of the traded stock, belongs to the minimal martingale measure, and not to the minimal entropy measure, as was originally derived in the reduced model of [41]. The above characterization of the minimal martingale measure is time-invariant. Since both the minimal martingale measure characterization and the form of the forward valuation algorithm are time-invariant, the forward indifference prices are independent of the forward normalization point, even though the forward dynamic utility itself is dependent on it. For the minimal entropy measure, the story is quite different. As our explicit characterization shows, the minimal entropy measure is dependent on the end of the investment horizon, at which time point the backward utility is normalized. As a result, even though the backward algorithm structure itself is time-invariant, the backward indifference prices are not. In addition, the characterization of the minimal martingale measure is simpler and more natural, rather than the minimal entropy one. Altogether, these factors make the forward dynamic utility a better candidate for valuing contracts affected by the non-traded risk factor.

We provided an explicit formula that shows how the backward and forward prices for European claims are related. We have shown that the backward price equals the difference of the two forward prices. The first is the price of an adjusted payoff, equal to contract's payoff less the entropy. The second is the price of a payoff equal to the entropy. Clearly, in a complete market the backward and forward prices are the same. We have shown that in the reduced model the two prices are the same as well. Also, in the reduced model the two measures, the minimal entropy measure and the minimal martingale measure coincide. Therefore both the backward and the forward utility pricing schemes are the direct extensions of the reduced model to a more general one.

We found that a number of continuous time results of [42] have their analogs in our binomial model. In particular, there is a clear analogy between the Sharpe ratio of the traded asset in continuous time and the local entropy terms in discrete time. This analogy starts with the structural similarity of the forward and backward utilities, and continues in the formulas for the aggregate entropy, and the reduced model condition. The continuous time model suggests that the entropy accumulates as a nonlinear functional of the integrated squared Sharpe ratio under the minimal martingale measure. This is similar to our discrete time model where the aggregate entropy is the nonlinear functional of the sum of local entropy terms. In the continuous model, if the Sharpe ratio is constant, then the backward and forward prices are the same. In discrete time, if the local entropy terms are constant, the two prices are the same as well. In addition, the continuous time model suggests pricing with forward utility via the nonlinear PDE associated with the minimal martingale measure, and the discrete time model suggests using the nonlinear pricing functional under same measure. We have also shown the above analogy between the discrete and continuous models for the backward utility and the minimal entropy measure.

The above similarities concern not only the European contracts, but also the early exercise contracts. We have shown that with both the backward and forward utilities in continuous time, there are values of the traded stock and the non-traded factor, for which the American price equals the intrinsic payoff of the contract. For other values of the traded stock and the non-traded risk factor, the European nonlinear PDE for the corresponding (backward or forward) indifference price holds. Similarly, in the discrete time model, for some values of S and Y, the price equals $C(S, Y)$. For others, the current indifference value is computed as the nonlinear functional of the next period's price. Another similarity we found is that the continuous time price of an American security is the supremum over all possible exercise times times of European prices of contracts with the maturity that corresponds to a particular stopping time. We have been able to derive the above result with both the forward and backward utility in discrete and continuous time.

For American contracts, in addition to the indifference value, we derived the optimal stopping time an the exercise policy. We have shown that it is optimal to exercise the American contract when the indifference value of alternative "to continue" crosses the level of the intrinsic payoff. The optimal hedging policy is the one that maximizes the next period's indifference value of the contract held, provided the contract will not be exercised immediately. With the forward utility, we have extended our algorithm to allow for partial exercise. Unlike in the complete market, since the non-traded stochastic factor affects the traded stock dynamics, partial exercise may be optimal. We provided a numerical example where indeed, only a fraction of all the contracts held is optimal to exercise.

In the non-reduced version of our binomial model, ξ^u_{t+1} and ξ^d_{t+1} are allowed to depend on the path of the stochastic factor up to time t. Conditional on time t, the time $t+1$ probability distribution of the traded asset is allowed to depend on the path of the non-traded risk factor up to time t as well. This makes our model a suitable application to stochastic volatility models, like Hull-White ([26]), Heston model ([23]), and other models. Our binomial model requires to discretize the original continuous time dynamics. For a simple continuous model, where the traded stock and the non-traded factor are both lognormal processes, the discretized version of the model has parameters ξ_t an η_t that ate constant, and the binomial tree implementation is straight forward, as examples of chapter 8 confirm. In a stochastic volatility model, the non-traded factor Y_t affects the value of ξ_{t+1}. As a result, the binomial tree built in a straight-forward way would not recombine. In this case the direct application of the binomial model to the state variables S and Y would lead to computationally intractable numerical procedure. A number of specialized ways for building recombining binomial trees for stochastic volatility models have been developed in the literature, including the works of [24], [43], and recently in [12]. The idea behind those papers in to transform existing state variables S and Y into new ones, and to generate paths of two new Brownian motions so that the binomial tree in the new variables recombines. The original state variables are then approximated via the inverse transformation of the new ones. The new, transformed variables may both depend on the non-traded risk factor, and our framework of one traded asset and one non-traded factor may not be applicable directly. The above indicates that more research is needed to be able to implement indifference pricing with stochastic volatility models in the discrete time model like the one we considered herein. The implementation using binomial trees is very intuitive and thus attractive, and has been used by industry practitioners for decades. Our discrete time algorithm definitely allows for an easy numerical implementation in a reduced model, as confirmed by examples provided herein. For the non-reduced model we have made the next necessary step of extending the theoretical derivations, creating a foundation for pricing via utility indifference in a more general class of models with binomial trees.

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